STRENGTH OF MATERIALS

DEPARTMENT OF MECHANICAL ENGINEERING

E.G.S.PILLAY ENGINEERING COLLEGE, NAGAPATTINAM

Details of Lecturer

- **Course Lecturer**:
  - Mr.B.MANIKANDAN (Asst. Professor)

COURSE GOALS

This course has two specific goals:

- (i) To introduce students to concepts of stresses and strain; shearing force and bending; as well as torsion and deflection of different structural elements.
- (ii) To develop theoretical and analytical skills relevant to the areas mentioned in (i) above.

COURSE OUTLINE

UNIT	TITLE	CONTENTS
I	DEFORMATION OF SOLIDS	Introduction to Rigid and Deformable bodies – properties, Stresses - Tensile, Compressive and Shear, Deformation of simple and compound bars under axial load – Thermal stress – Elastic constants – Volumetric Strain, Strain energy and unit strain energy
II	TORSION	Introduction - Torsion of Solid and hollow circular bars – Shear stress distribution – Stepped shaft – Twist and torsion stiffness – Compound shafts – Springs – types - helical springs – shear stress and deflection in springs
III	BEAMS	Types : Beams , Supports and Loads – Shear force and Bending Moment – Cantilever, Simply supported and Overhanging beams – Stresses in beams – Theory of simple bending – Shear stresses in beams – Evaluation of 'I', 'C' & 'T' sections

COURSE OUTLINE

UNIT	TITLE	CONTENTS
IV	DEFLECTION OF BEAMS	Introduction - Evaluation of beam deflection and slope: Macaulay Method and Moment-area Method
V	ANALYSIS OF STRESSES IN TWO DIMENSIONS	Biaxial state of stresses – Thin cylindrical and spherical shells – Deformation in thin cylindrical and spherical shells – Principal planes and stresses – Mohr's circle for biaxial stresses – Maximum shear stress - Strain energy in bending and torsion

TEXT BOOKS

- Bansal, R.K., *A Text Book of Strength of Materials*, Lakshmi Publications Pvt. Limited, New Delhi, 1996
- Ferdinand P.Beer, and Rusell Johnston, E., *Mechanics of Materials*, SI Metric Edition, McGraw Hill, 1992

Course Objectives

Upon successful completion of this course, students should be able to:

- (i) Understand and solve simple problems involving stresses and strain in two and three dimensions.
- (ii) Understand the difference between statically determinate and indeterminate problems.
- (iv) Analyze stresses in two dimensions and understand the concepts of principal stresses and the use of Mohr circles to solve two-dimensional stress problems.

COURSE OBJECTIVES CONTD.

- (v) Draw shear force and bending moment diagrams of simple beams and understand the relationships between loading intensity, shearing force and bending moment.
- (vi) Compute the bending stresses in beams with one or two materials.
- (vii) Calculate the deflection of beams using the direct integration and moment-area method.

Teaching Strategies

- The course will be taught via Lectures. Lectures will also involve the solution of tutorial questions. Tutorial questions are designed to complement and enhance both the lectures and the students appreciation of the subject.
- Course work assignments will be reviewed with the students.

UNITS:

	British	Metric	S.I.
1. Force	Ib, kip, Ton	g, kg,	N, kN
	1 kip = 1000 Ib 1 ton = 2240 Ib	1 kg = 1000 g Ton = 1000 kg	1 kN = 1000 N 1 kg = 10 N
2. Long	in, ft	m, cm, mm	m, cm, mm
	1 f = 12 in	1 m = 100 cm 1 cm = 10 mm 1 m =1000 mm 1 in = 2.54 cm	1 m = 100 cm 1 cm = 10 mm 1 m =1000 mm 1 in = 2.54 cm
3. Stress	psi, ksi $\frac{p}{in^2}, \frac{kip}{in^2}$	Pa ($\frac{N}{mm^2}$), MPa, GPa	

$$MPa = 10^6\ Pa = 10^6\ N/mm^2 \times \frac{1}{1000^2 \frac{mm^2}{m^2}}$$

$$MPa = \frac{N}{mm^2}$$

$$GPa = 10^9\ Pa = 10^9\ N/mm^2 \times \frac{1}{1000^2 \frac{mm^2}{m^2}} = 10^3 \frac{N}{mm^2} \times \frac{1}{1000 \frac{N}{kN}}$$

$$GPa = kN/mm^2$$

UNIT I

STRESS AND STRAIN RELATIONS

DIRECT OR NORMAL STRESS

- When a force is transmitted through a body, the body tends to change its shape or deform. The body is said to be strained.

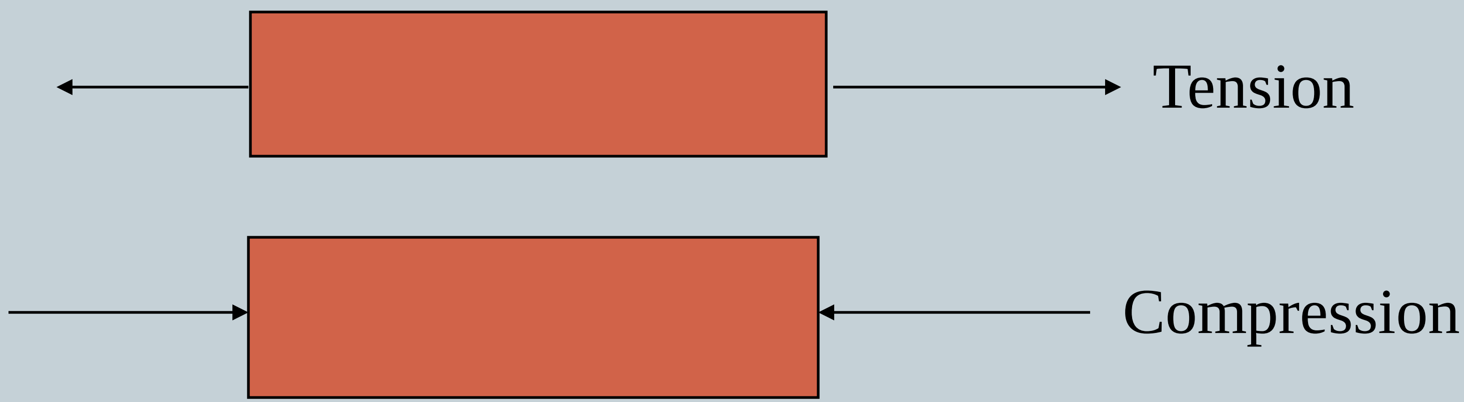
- Direct Stress = Applied Force (F) / Cross Sectional Area (A)

$$\text{Direct Stress} = \frac{\text{Applied Force (F)}}{\text{Cross Sectional Area (A)}}$$

- **Units:** Usually N/m^2 (Pa), N/mm^2, MN/m^2, GN/m^2 or N/cm^2
- **Note:** $1\ N/mm^2 = 1\ MN/m^2 = 1\ MPa$

Direct Stress Contd.

- Direct stress may be tensile or compressive and result from forces acting perpendicular to the plane of the cross-section



Tension and Compression

Direct or Normal Strain

- When loads are applied to a body, some deformation will occur resulting to a change in dimension.
- Consider a bar, subjected to axial tensile loading force, F. If the bar extension is dl and its original length (before loading) is L, then tensile strain is:

Direct Strain (ε) = Change in Length / Original Length

$$\text{Direct Strain } (\varepsilon) = \frac{\text{Change in Length}}{\text{Original Length}}$$

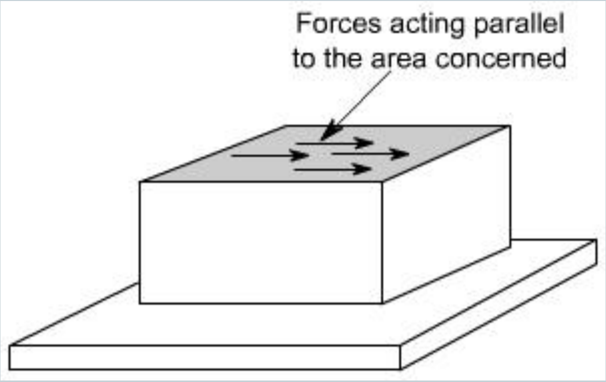
i.e. $\varepsilon = dl/L$

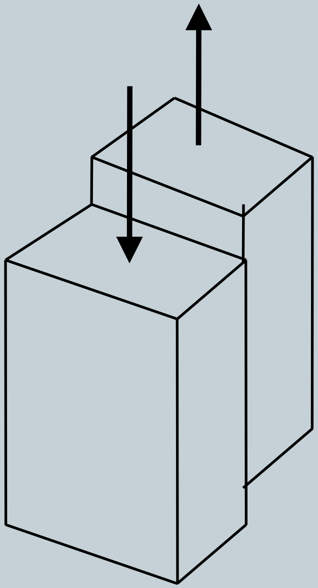
Direct or Normal Strain Contd.

- As strain is a ratio of lengths, it is dimensionless.
- Similarly, for compression by amount, dl: Compressive strain = - dl/L
- **Note:** Strain is positive for an increase in dimension and negative for a reduction in dimension.

Shear Stress and Shear Strain

- Shear stresses are produced by equal and opposite parallel forces not in line.
- The forces tend to make one part of the material slide over the other part.
- Shear stress is tangential to the area over which it acts.





Ultimate Strength

The **strength** of a material is a measure of the stress that it can take when in use. The **ultimate strength** is the measured stress at failure but this is not normally used for design because safety factors are required. The normal way to define a safety factor is :

$$\text{safety factor} = \frac{\text{stress at failure}}{\text{stress when loaded}} = \frac{\text{Ultimate stress}}{\text{Permissible stress}}$$

Strain

We must also define **strain**. In engineering this is not a measure of force but is a measure of the deformation produced by the influence of stress. For tensile and compressive loads:

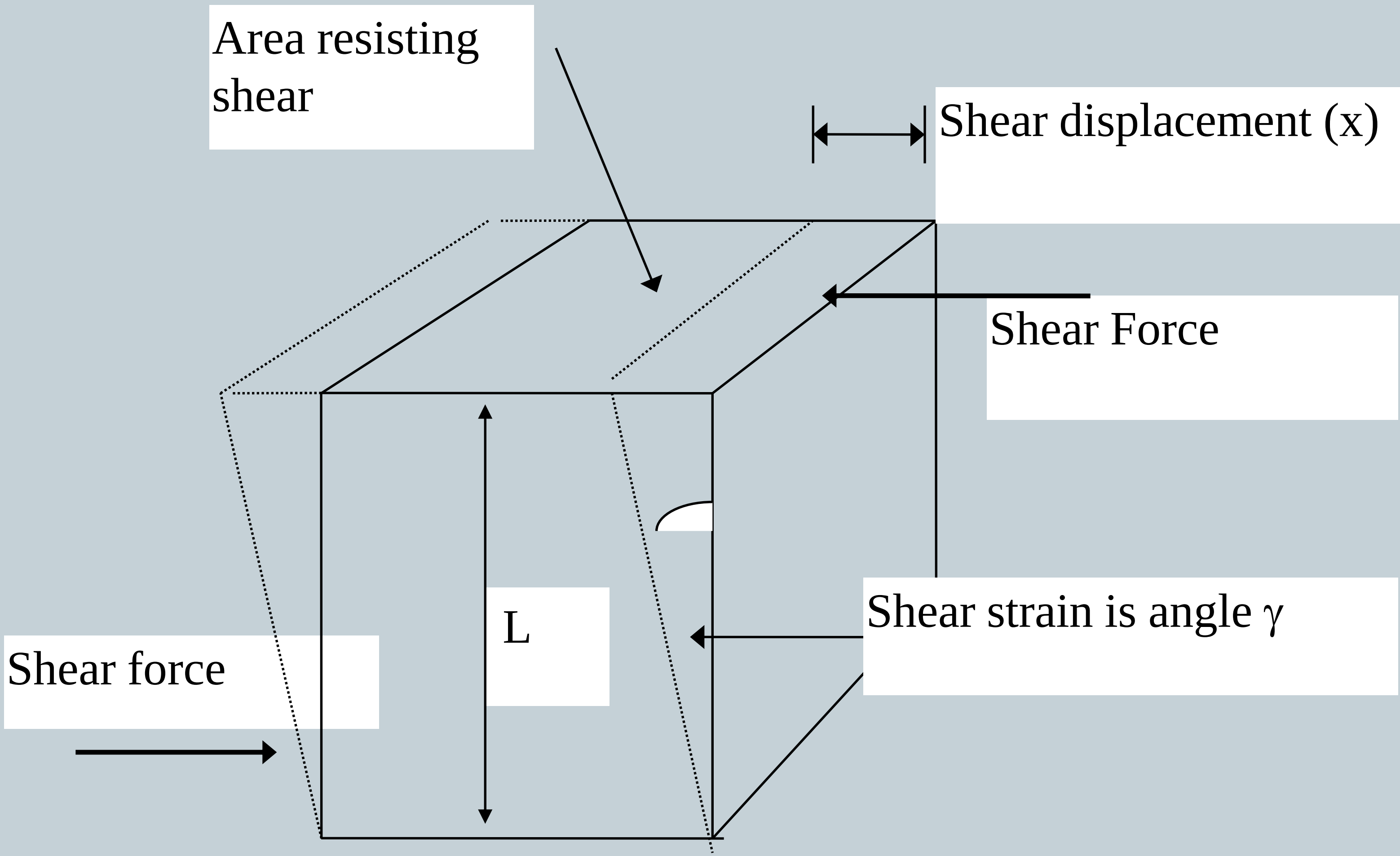
$$strain \quad \varepsilon = \frac{increase\ in\ length\ \ x}{original\ length\ \ L}$$

Strain is dimensionless, i.e. it is not measured in metres, killogrammes etc.

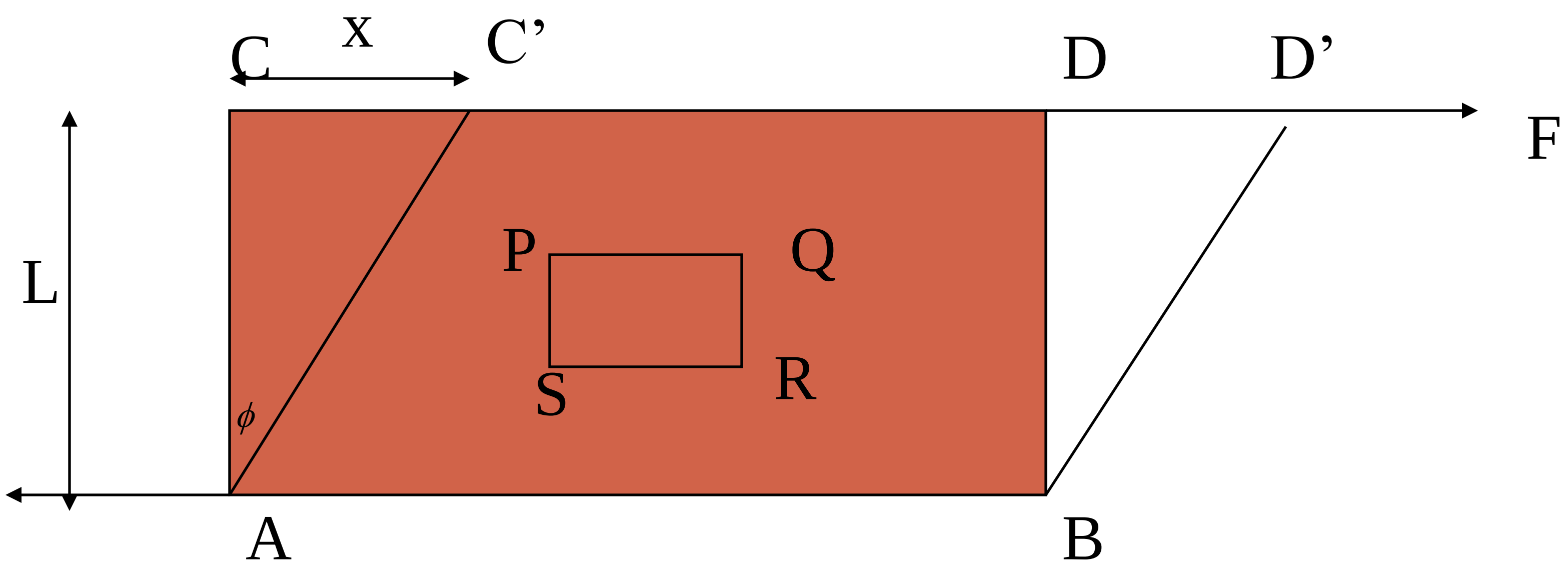
$$shear\ strain \quad \gamma \approx \frac{shear\ displacement\ \ x}{width\ \ L}$$

For shear loads the strain is defined as the angle γ This is measured in radians

Shear stress and strain



Shear Stress and Shear Strain Contd.



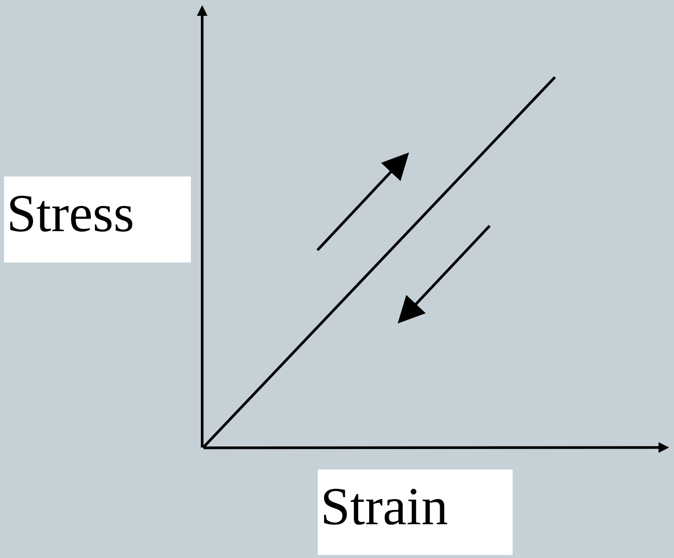
Shear strain is the distortion produced by shear stress on an element or rectangular block as above. The shear strain, γ (gamma) is given as:

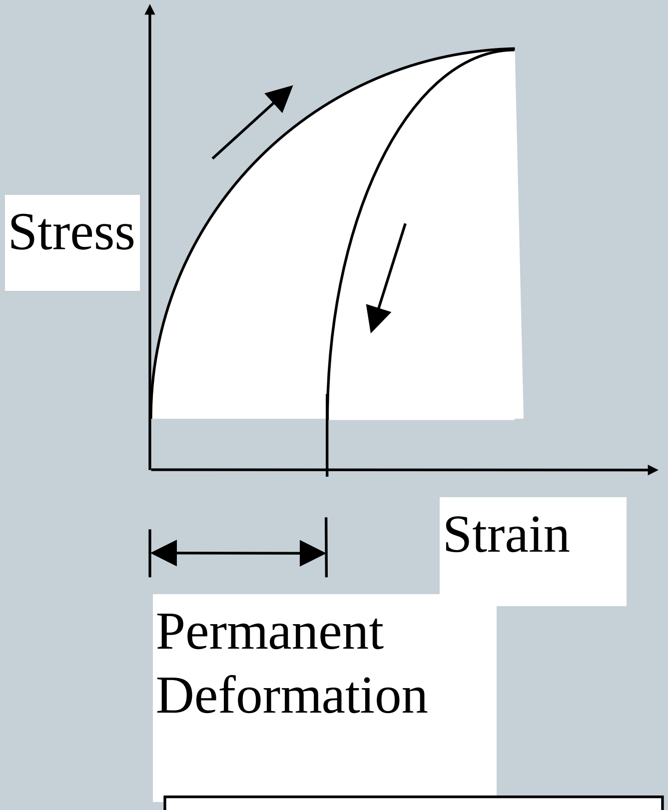
$$\gamma = x/L = \tan\phi$$

Shear Stress and Shear Strain Concluded

- For small , ϕ $\gamma = \phi$
- Shear strain then becomes the change in the right angle.
- It is dimensionless and is measured in radians.

Elastic and Plastic deformation





Elastic deformation

Plastic deformation

Modulus of Elasticity

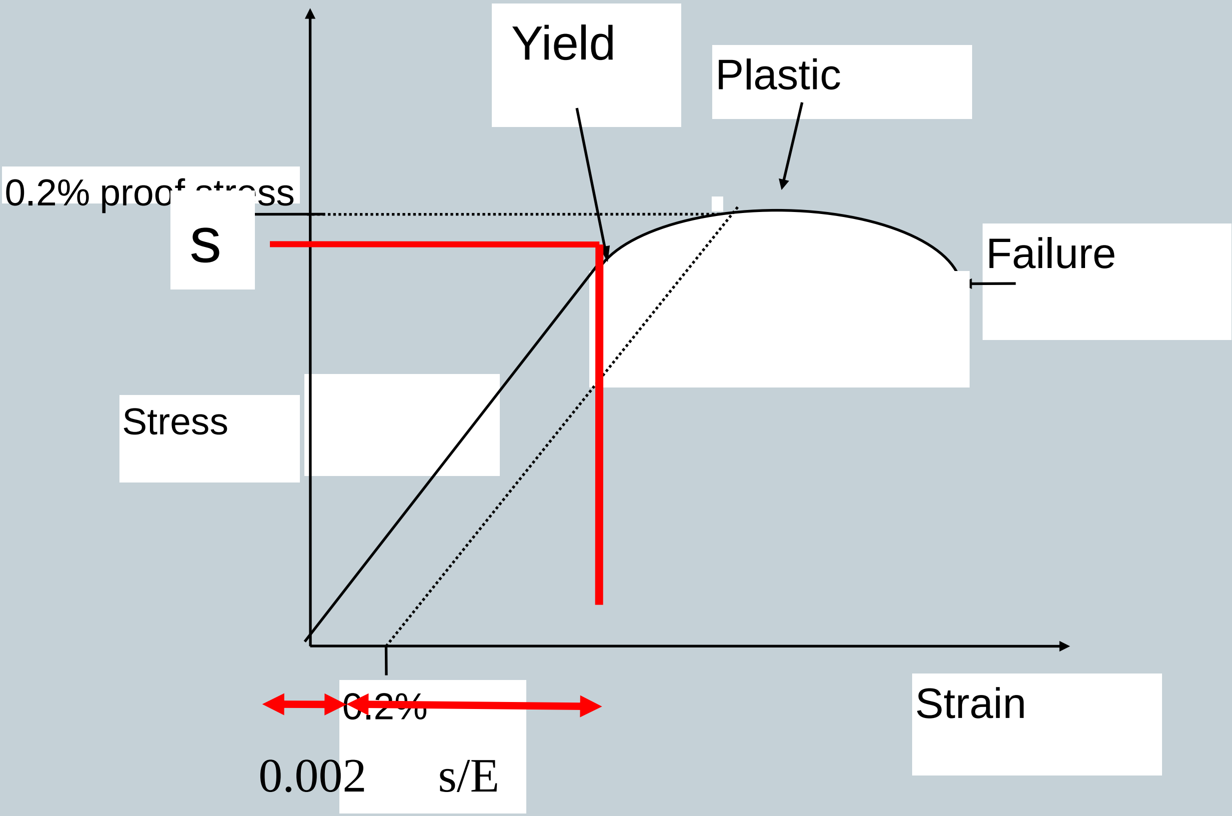
If the strain is "elastic" Hooke's law may be used to define

$$\text{Youngs Modulus} \quad E = \frac{\text{Stress}}{\text{Strain}} = \frac{W}{x} \times \frac{L}{A}$$

Young's modulus is also called the modulus of elasticity or stiffness and is a measure of how much strain occurs due to a given stress. Because strain is dimensionless Young's modulus has the units of stress or pressure

How to calculate deflection if the proof stress is applied and then partially removed.

If a sample is loaded up to the 0.2% proof stress and then unloaded to a stress s the strain x = 0.2% + s/E   where E is the Young's modulus



Volumetric Strain

- Hydrostatic stress refers to tensile or compressive stress in all dimensions within or external to a body.
- Hydrostatic stress results in change in volume of the material.
- Consider a cube with sides x, y, z. Let dx, dy, and dz represent increase in length in all directions.
- i.e. new volume = $(x + dx)\ (y + dy)\ (z + dz)$

Volumetric Strain Contd.

- Neglecting products of small quantities:
- New volume = x y z + z y dx + x z dy + x y dz
- Original volume = x y z
- = z y dx + x z dy + x y dz
- Volumetric strain ΔV = z y dx + x z dy + x y dz / x y z ε_v
- ε_v = dx/x + dy/y + dz/z

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Elasticity and Hooke's Law

- All solid materials deform when they are stressed, and as stress is increased, deformation also increases.
- If a material returns to its original size and shape on removal of load causing deformation, it is said to be **elastic**.
- If the stress is steadily increased, a point is reached when, after the removal of load, not all the induced strain is removed.
- This is called the elastic limit.

Hooke's Law

- States that providing the limit of proportionality of a material is not exceeded, the stress is directly proportional to the strain produced.
- If a graph of stress and strain is plotted as load is gradually applied, the first portion of the graph will be a straight line.
- The slope of this line is the constant of proportionality called modulus of Elasticity, E or Young's Modulus.
- It is a measure of the stiffness of a material.

Hooke's Law

Modulus of Elasticity, E = $\dfrac{Direct\ stress}{Direct\ strain} = \dfrac{\sigma}{\varepsilon}$

Also: For Shear stress: Modulus of rigidity or shear modulus, G = $\dfrac{Shear\ stress}{Shear\ strain} = \dfrac{\tau}{\gamma}$

Also: Volumetric strain, ε_v is proportional to hydrostatic stress, σ within the elastic range

i.e. : $\sigma / \varepsilon_v = K$ called **bulk modulus**.

Stress-Strain Relations of Mild Steel

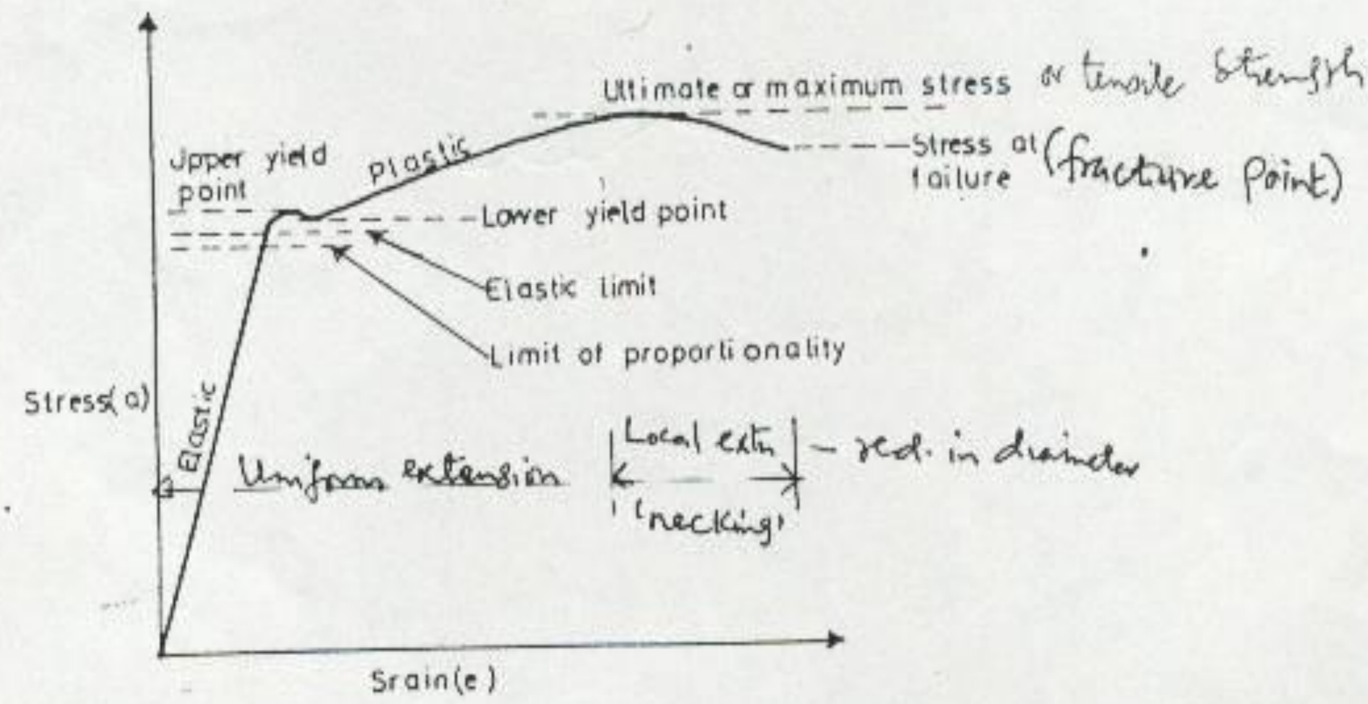


Fig: Behaviour of mild-steel rod under tension.

Equation For Extension

From the above equations:

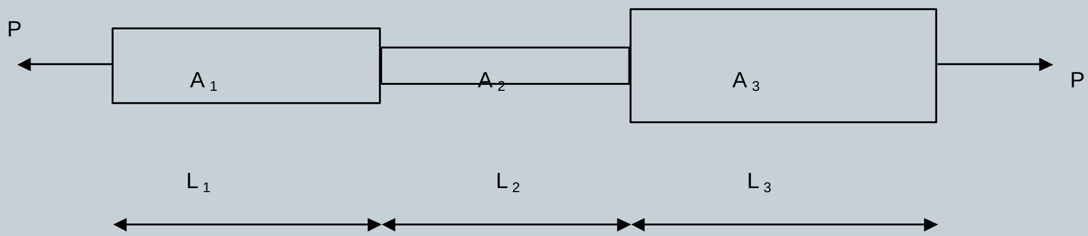
$$E = \frac{\sigma}{\varepsilon} = \frac{F/A}{dl/L} = \frac{F\,L}{A\,dl}$$

$$dl = \boxed{\frac{F\,L}{A\,E}}$$

This equation for extension is very important

Extension For Bar of Varying Cross Section

For a bar of varying cross section:



$$dl = \frac{F}{E}\left[\frac{L_1}{A_1} + \frac{L_2}{A_2} + \frac{L_3}{A_3}\right]$$

Factor of Safety

- The load which any member of a machine carries is called working load, and stress produced by this load is the working stress.
- Obviously, the working stress must be less than the yield stress, tensile strength or the ultimate stress.
- This working stress is also called the permissible stress or the allowable stress or the design stress.

Factor of Safety Contd.

- Some reasons for factor of safety include the inexactness or inaccuracies in the estimation of stresses and the non-uniformity of some materials.

$$\text{Factor of safety} = \frac{\textit{Ultimate or yield stress}}{\textit{Design or working stress}}$$

Note: Ultimate stress is used for materials e.g. concrete which do not have a well-defined yield point, or brittle materials which behave in a linear manner up to failure. Yield stress is used for other materials e.g. steel with well defined yield stress.

Results From a Tensile Test

(a) Modulus of Elasticity, $E = \dfrac{Stress\ up\ to\ limit\ of\ proportionality}{Strain}$

(b) Yield Stress or Proof Stress (See below)

(c) Percentage elongation = $\dfrac{Increase\ in\ gauge\ length}{Original\ gauge\ length} \times 100$

(d) Percentage reduction in area = $\dfrac{Original\ area - area\ at\ fracture}{Original\ area} \times 100$

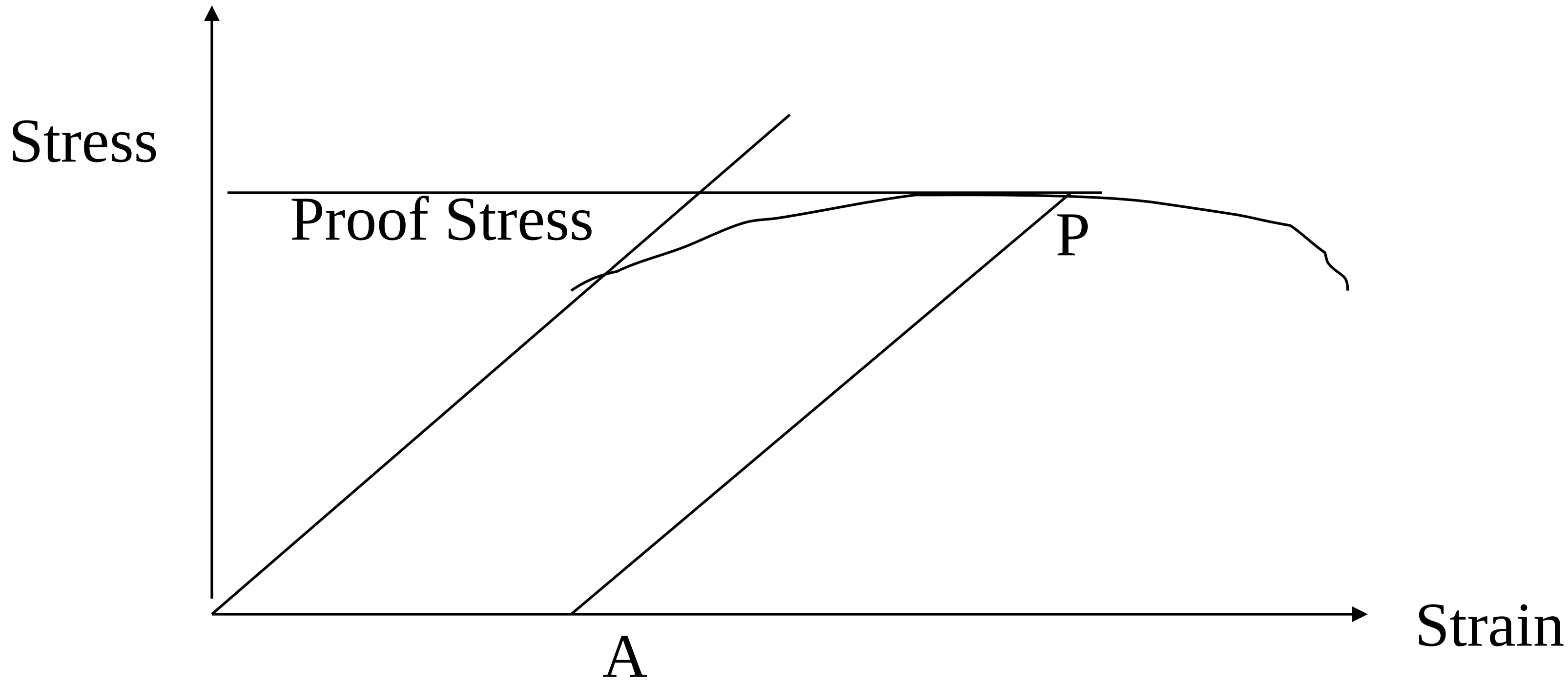
(e) Tensile Strength = $\dfrac{Maximum\ load}{Original\ cross\ sectional\ area}$

The percentage of elongation and percentage reduction in area give an indication of the ductility of the material i.e. its ability to withstand strain without fracture occurring.

Proof Stress

- High carbon steels, cast iron and most of the non-ferrous alloys do not exhibit a well defined yield as is the case with mild steel.
- For these materials, a limiting stress called proof stress is specified, corresponding to a non-proportional extension.
- The non-proportional extension is a specified percentage of the original length e.g. 0.05, 0.10, 0.20 or 0.50%.

Determination of Proof Stress



The proof stress is obtained by drawing AP parallel to the initial slope of the stress/strain graph, the distance, OA being the strain corresponding to the required non-proportional extension e.g. for 0.05% proof stress, the strain is 0.0005.

Thermal Strain

Most structural materials expand when heated,

in accordance to the law: $\varepsilon = \alpha \ T$

where ε is linear strain and

α is the coefficient of linear expansion;

T is the rise in temperature.

That is for a rod of Length, L;

if its temperature increased by t, the extension,

dl = α L T.

Thermal Strain Contd.

As in the case of lateral strains, thermal strains do not induce stresses unless they are constrained.

The total strain in a body experiencing thermal stress may be divided into two components:

Strain due to stress, ε_{σ} and

That due to temperature, ε_{T} .

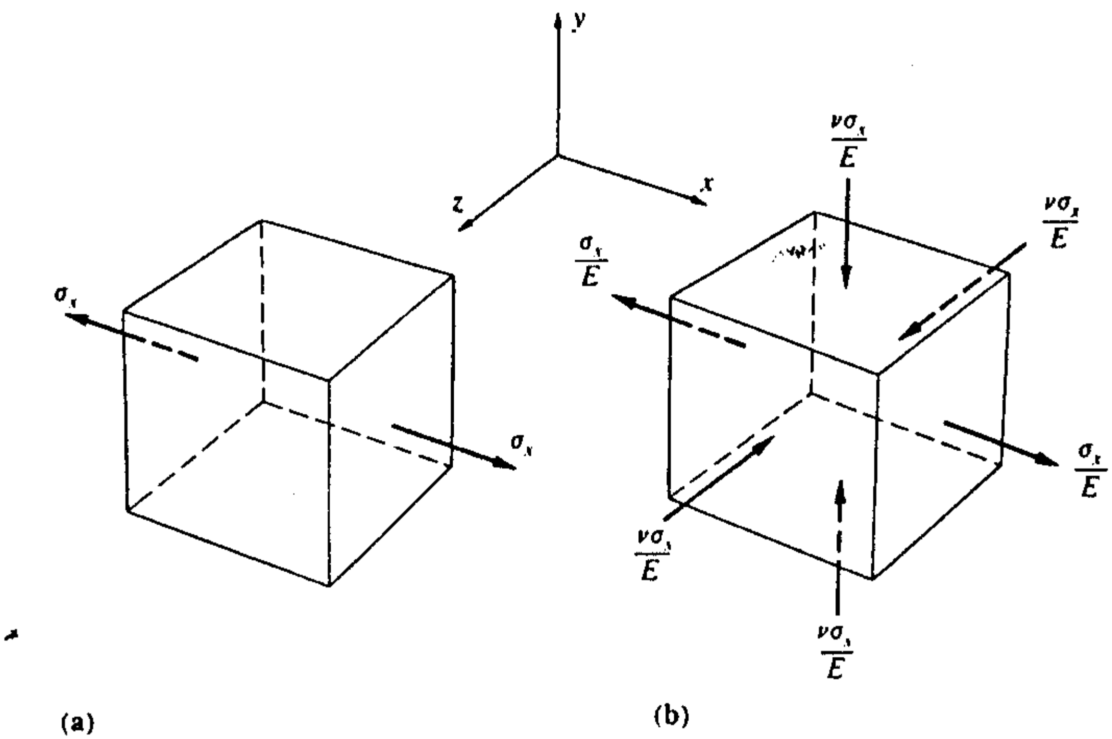
Thus: $\varepsilon = \varepsilon_{\sigma} + \varepsilon_{T}$

$$\varepsilon = \frac{\sigma}{E} + \alpha T$$

Principle of Superposition

- It states that the effects of several actions taking place simultaneously can be reproduced exactly by adding the effect of each action separately.
- The principle is general and has wide applications and holds true if:
- (i) The structure is elastic
- (ii) The stress-strain relationship is linear
- (iii) The deformations are small.

General Stress-Strain Relationships



Relationship between Elastic Modulus (E) and Bulk Modulus, K

It has been shown that: $\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$

$$\varepsilon_x = \frac{1}{E}\left|\sigma_x - \upsilon(\sigma_y + \sigma_z)\right|$$

For hydrostatic stress, $\sigma_x = \sigma_y = \sigma_z = \sigma$

$$i.e. \quad \varepsilon_x = \frac{1}{E}\left|\sigma - 2\sigma\upsilon\right| = \frac{\sigma}{E}\left|1 - 2\upsilon\right|$$

Similarly, ε_y *and* ε_z *are each* $\frac{\sigma}{E}\left|1 - 2\upsilon\right|$

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z = \text{Volumetric strain}$$

$$\varepsilon_v = \frac{3\sigma}{E}\left|1 - 2\upsilon\right|$$

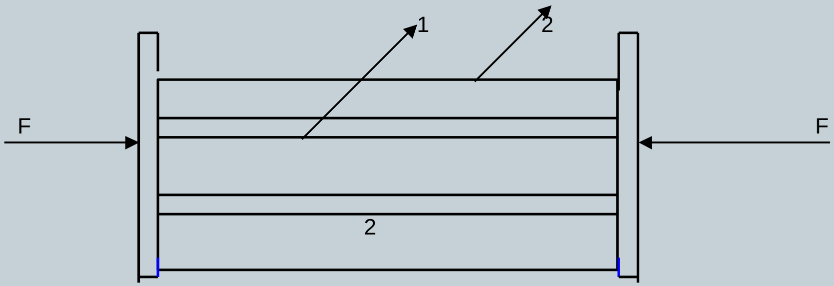
$$E = \frac{3\sigma}{\varepsilon_v}\left|1 - 2\upsilon\right|$$

$$\text{Bulk Modulus, } K = \frac{\text{Volumetric or hydrostatic stress}}{\text{Volumetric strain}} = \frac{\sigma}{\varepsilon_v}$$

$$i.e. \quad E = 3K\left|1 - 2\upsilon\right| \quad \text{and} \quad K = \frac{E}{3\left|1 - 2\upsilon\right|}$$

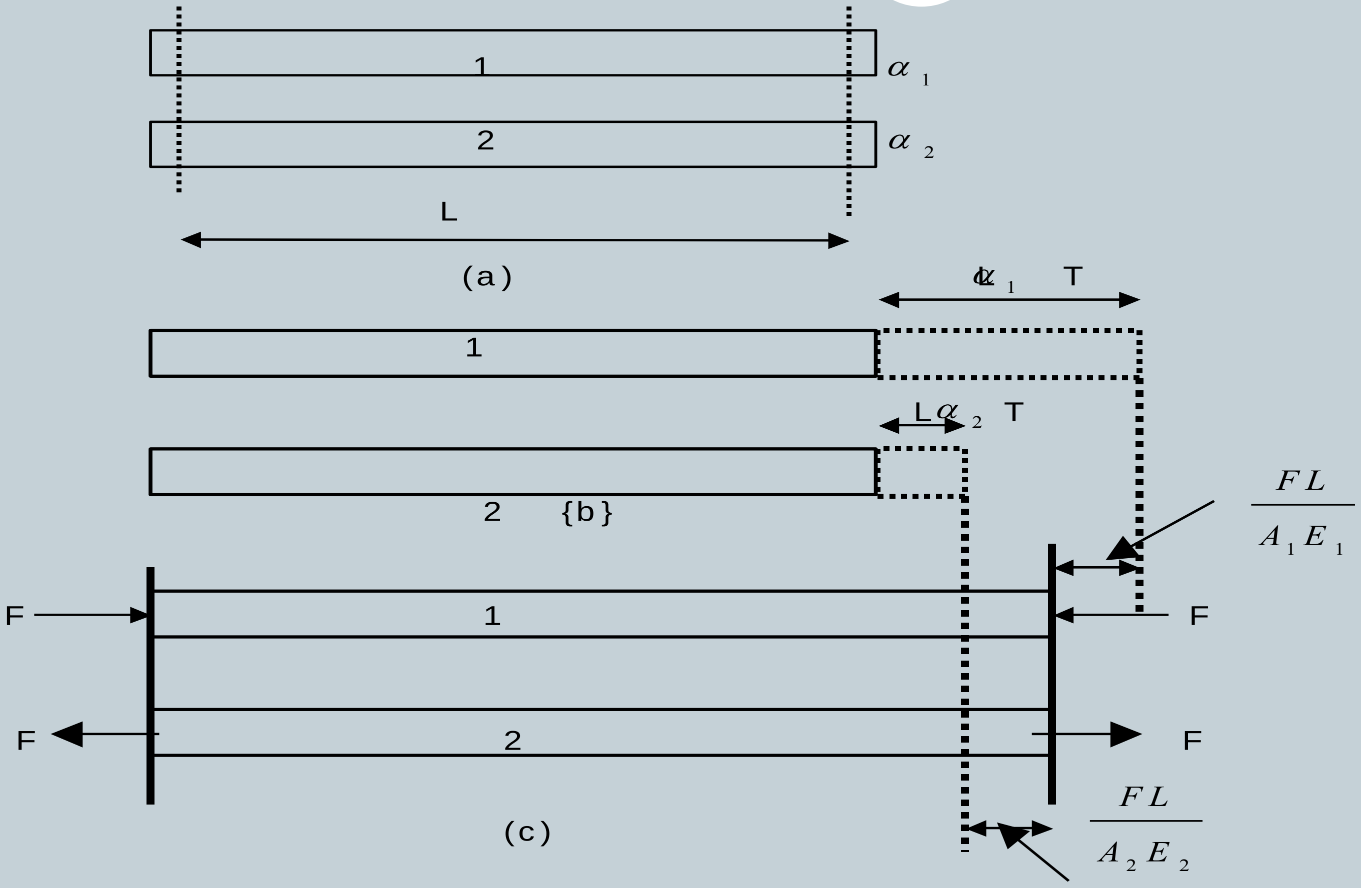
Compound Bars

A compound bar is one comprising two or more parallel elements, of different materials, which are fixed together at their end. The compound bar may be loaded in tension or compression.



Section through a typical compound bar consisting of a circular bar (1) surrounded by a tube (2)

Temperature stresses in compound bars

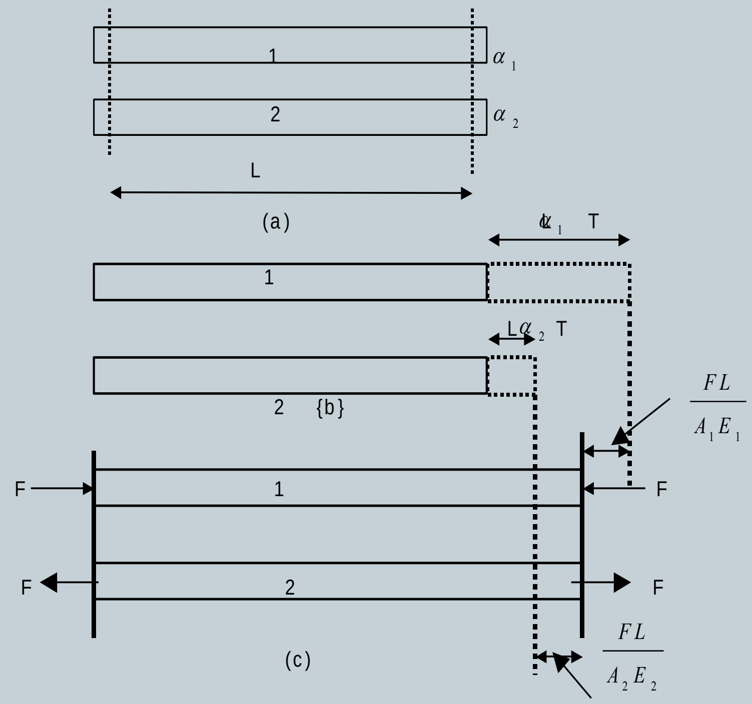


Temperature Stresses Contd.

Free expansions in bars (1) and (2) are $L\alpha_1 T$ *and* $L\alpha_2 T$ respectively.

Due to end fixing force, F: the decrease in length of bar (1) is

$\frac{FL}{A_1E_1}$ and the increase in length of (2) is $\frac{FL}{A_2E_2}$.



At Equilibrium:

$$L\alpha_1 T - \frac{FL}{A_1E_1} = L\alpha_2 T + \frac{FL}{A_2E_2}$$

$$i.e.\quad F\left[\frac{1}{A_1E_1} + \frac{1}{A_2E_2}\right] = T(\alpha_1 - \alpha_2)$$

$$i.e.\quad \sigma_1 A_1 \left[\frac{A_2E_2 + A_1E_1}{E_1E_2A_1A_2}\right] = T(\alpha_1 - \alpha_2)$$

$$\boxed{\sigma_1 = \frac{T(\alpha_1 - \alpha_2)A_2E_1E_2}{A_1E_1 + A_2E_2}}$$

$$\boxed{\sigma_2 = \frac{T(\alpha_1 - \alpha_2)A_1E_1E_2}{A_1E_1 + A_2E_2}}$$

Note: As a result of Force, F, bar (1) will be in compression while (2) will be in tension.

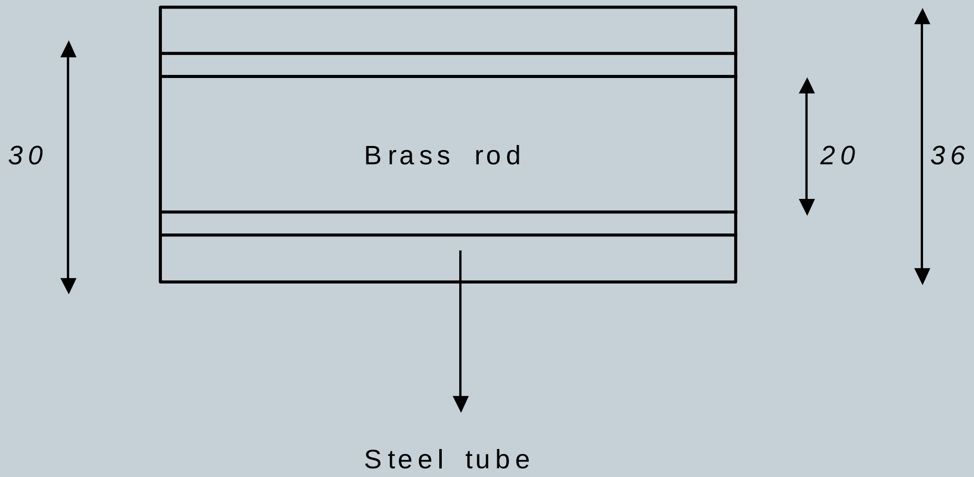
Example

- A steel tube having an external diameter of 36 mm and an internal diameter of 30 mm has a brass rod of 20 mm diameter inside it, the two materials being joined rigidly at their ends when the ambient temperature is 18 ^{0}C. Determine the stresses in the two materials: (a) when the temperature is raised to 68 ^{0}C (b) when a compressive load of 20 kN is applied at the increased temperature.

Example Contd.

- For brass: Modulus of elasticity = 80 GN/m^2; Coefficient of expansion = 17 x 10 -6 /^{0}C
- For steel: Modulus of elasticity = 210 GN/m^2; Coefficient of expansion = 11 x 10 -6 /^{0}C

Solution



Area of brass rod (A_b) = $\dfrac{\pi \ x\ 20^2}{4} = 314.16\ mm^2$

Area of steel tube (A_s) = $\dfrac{\pi \ x\ (36^2 - 30^2)}{4} = 311.02\ mm^2$

$$A_s E_s = 311.02\ x\ 10^{-6} m^2\ x\ 210\ x\ 10^9\ N/m^2 = 0.653142\ x\ 10^8\ N$$

$$\frac{1}{A_s E_s} = 1.53106\ x\ 10^{-8}$$

Solution Contd.

$$A_b E_b = 314.16 \ x \ 10^{-6} m^2 \ x \ 80 \ x \ 10^9 N / m^2 = 0.251327 \ x \ 10^8 N$$

$$\frac{1}{A_b E_b} = 3.9788736 \ x \ 10^{-8}$$

$$T(\alpha_b - \alpha_s) = 50(17 - 11) \ x \ 10^{-6} = 3 \ x \ 10^{-4}$$

With increase in temperature, brass will be in compression while steel will be in tension. This is because expands more than steel.

$$i.e. \quad F\left[\frac{1}{A_s E_s} + \frac{1}{A_b E_b}\right] = T(\alpha_b - \alpha_s)$$

i.e. $F[1.53106 + 3.9788736] \times 10^{-8} = 3 \times 10^{-4}$

F = 5444.71 N

Solution Concluded

Stress in steel tube = $\dfrac{5444.71\,N}{311.02\ mm^2} = 17.51\,N/mm^2 = 17.51\,MN/m^2\,(Tension)$

Stress in brass rod = $\dfrac{5444.71\,N}{314.16\ mm^2} = 17.33\,N/mm^2 = 17.33\,MN/m^2\,(Compression)$

(b) Stresses due to compression force, F' of 20 kN

$$\sigma_s = \frac{F'E_s}{E_sA_s + E_bA_b} = \frac{20\ x\ 10^3\,N\ \ x\ 210\ x\ 10^9\,N/m^2}{0.653142 + 0.251327\ x\ 10^8} = 46.44\,MN/m^2\,(Compression)$$

$$\sigma_b = \frac{F'E_b}{E_sA_s + E_bA_b} = \frac{20\ x\ 10^3\,N\ \ x\ 80\ x\ 10^9\,N/m^2}{0.653142 + 0.251327\ x\ 10^8} = 17.69\,MN/m^2\,(Compression)$$

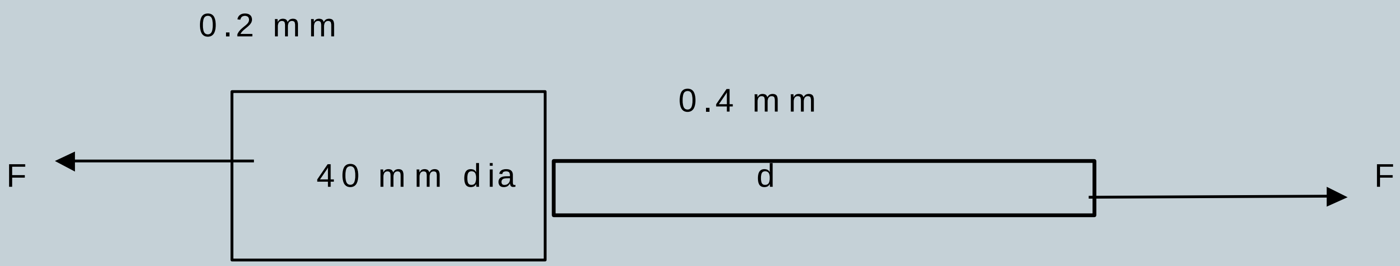
Resultant stress in steel tube = - 46.44 + 17.51 = 28.93 MN/m^2 (Compression)

Resultant stress in brass rod = -17.69 - 17.33 = 35.02 MN/m^2 (Compression)

Example

- A composite bar, 0.6 m long comprises a steel bar 0.2 m long and 40 mm diameter which is fixed at one end to a copper bar having a length of 0.4 m.
- Determine the necessary diameter of the copper bar in order that the extension of each material shall be the same when the composite bar is subjected to an axial load.
- What will be the stresses in the steel and copper when the bar is subjected to an axial tensile loading of 30 kN? (For steel, E = 210 GN/m^2; for copper, E = 110 GN/m^2)

Solution



Let the diameter of the copper bar be d mm

Specified condition: Extensions in the two bars are equal

$$dl_c = dl_s$$

$$dl = \varepsilon L = \frac{\sigma}{E} L = \frac{FL}{AE}$$

Thus: $$\frac{F_c L_c}{A_c E_c} = \frac{F_s L_s}{A_s E_s}$$

Solution Concluded

Also: Total force, F is transmitted by both copper and steel

i.e. F_c = Fs = F

$$i.e. \quad \frac{L_c}{A_c E_c} = \frac{L_s}{A_s E_s}$$

Substitute values given in problem:

$$\frac{0.4\ m}{\pi\ d^2/4\ m^2\ 110\ x\ 10^9\ N/m^2} = \frac{0.2\ m}{\pi/4\ x\ 0.040^2\ x\ 210\ x\ 10^9\ N/m^2}$$

$$d^2 = \frac{2\ x\ 210\ x\ 0.040^2}{110} m^2; \quad d = 0.07816\ m = 78.16\ mm.$$

Thus for a loading of 30 kN

Stress in steel, $\sigma_s = \dfrac{30\ x\ 10^3\ N}{\pi/4\ x\ 0.040^2\ x\ 10^{-6}} = 23.87\ MN/m^2$

Stress in copper, $\sigma_c = \dfrac{30\ x\ 10^3\ N}{\pi/4\ x\ 0.07816^2\ x\ 10^{-6}} = 9\ MN/m^2$

Elastic Strain Energy

- If a material is strained by a gradually applied load, then work is done on the material by the applied load.
- The work is stored in the material in the form of strain energy.
- If the strain is within the elastic range of the material, this energy is not retained by the material upon the removal of load.

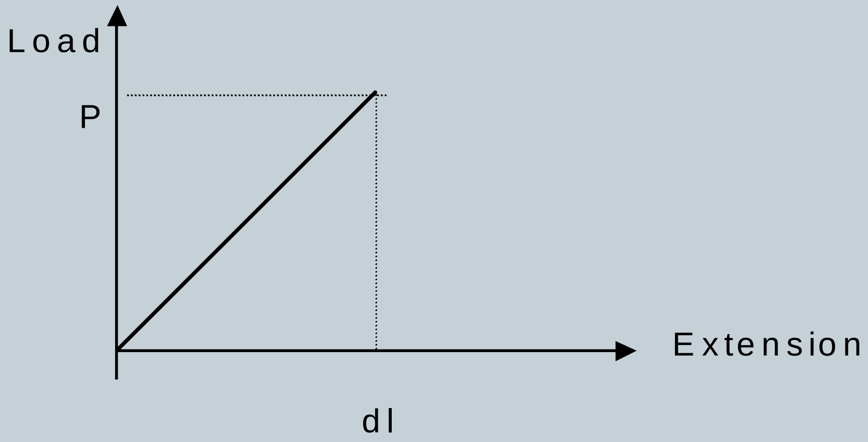
Elastic Strain Energy Contd.

Figure below shows the load-extension graph of a uniform bar.

The extension dl is associated with a gradually applied load, P

which is within the elastic range. The shaded area represents

the work done in increasing the load from zero to its value



Work done = strain energy of bar = shaded area

Elastic Strain Energy Concluded

$W = U = 1/2\ P\ dl$ (1)

Stress, $\sigma = P/A$ i.e $P = \sigma A$

Strain = Stress/E

i.e $dl/L = \sigma/E$, $dl = (\sigma L)/E$ L= original length

Substituting for P and dl in Eqn (1) gives:

$W = U = 1/2\ \sigma A . (\sigma L)/E = \sigma^2/2E \times A\ L$

A L is the volume of the bar.

i.e $U = \sigma^2/2E \times \text{Volume}$

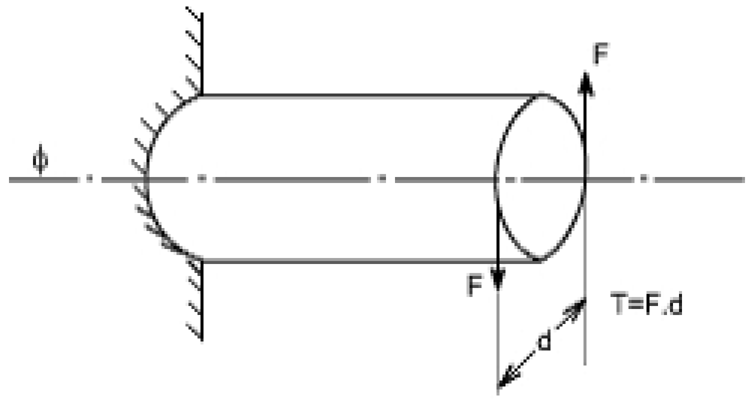
The units of strain energy are same as those of work i.e. Joules. Strain energy per unit volume, $\sigma^2/2E$ is known as resilience. The greatest amount of energy that can stored in a material without permanent set occurring will be when σ is equal to the elastic limit stress.

UNIT 2

TORSION

Torsion of circular shafts

Definition of Torsion: Consider a shaft rigidly clamped at one end and twisted at the other end by a torque T = F.d applied in a plane perpendicular to the axis of the bar such a shaft is said to be in torsion.



Effects of Torsion: The effects of a torsional load applied to a bar are

(i) To impart an angular displacement of one end cross – section with respect to the other end.

(ii) To setup shear stresses on any cross section of the bar perpendicular to its axis.

Assumption:

(i) The materiel is homogenous i.e of uniform elastic properties exists throughout the material.

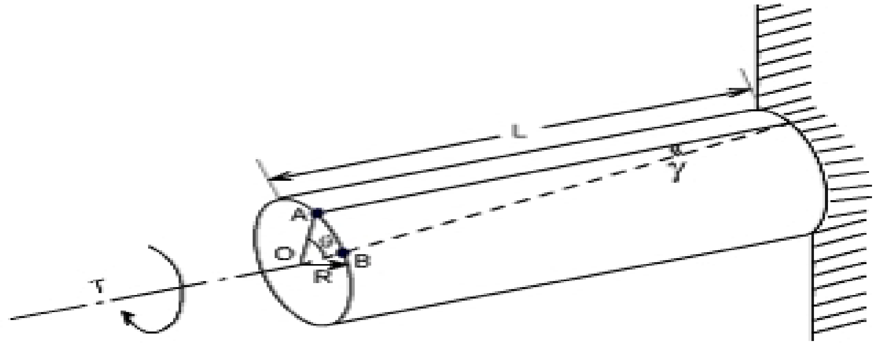
(ii) The material is elastic, follows Hook's law, with shear stress proportional to shear strain.

(iii) The stress does not exceed the elastic limit.

(iv) The circular section remains circular

(v) Cross section remain plane.

(vi) Cross section rotate as if rigid i.e. every diameter rotates through the same angle.



Consider now the solid circular shaft of radius R subjected to a torque T at one end, the other end being fixed Under the action of this torque a radial line at the free end of the shaft twists through an angle q , point A moves to B, and AB subtends an angle ' g ' at the fixed end. This is then the angle of distortion of the shaft i.e the shear strain.

Since angle in radius = arc / Radius

arc AB = Rq

= L g [since L and g also constitute the arc AB]

Thus, g = Rq / L (1)

From the definition of Modulus of rigidity or Modulus of elasticity in shear

$$G = \frac{\text{shear stress}(\tau)}{\text{shear strain}(\gamma)}$$

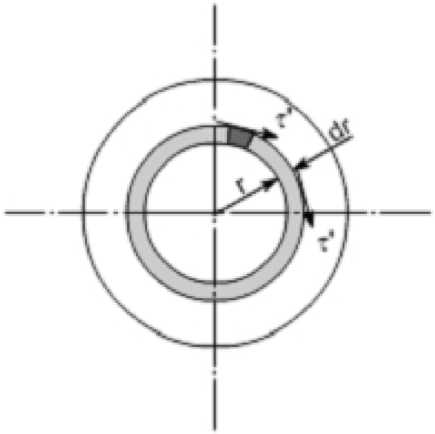
where γ is the shear stress set up at radius R.

Then $\frac{\tau}{G} = \gamma$

Equating the equations (1) and (2) we get $\frac{R\theta}{L} = \frac{\tau}{G}$

$\frac{\tau}{R} = \frac{G\theta}{L}\left(= \frac{\tau'}{r}\right)$ where τ' is the shear stress at any radius r.

Stresses: Let us consider a small strip of radius r and thickness dr which is subjected to shear stress t'.



The force set up on each element

= stress x area

= t' x 2p r dr (approximately)

This force will produce a moment or torque about the center axis of the shaft.

= t' . 2 p r dr . r

= 2 p t' . r^2. dr

The total torque T on the section, will be the sum of all the contributions. $T = \int_0^R 2\pi\tau' r^2 \, dr$

Since t' is a function of r, because it varies with radius so writing down t' in terms of r from the equation (1).

i.e $\tau' = \frac{G\theta . r}{L}$

we get $T = \int_0^R 2\pi \frac{G\theta}{L} . r^3 \, dr$

$$T = \frac{2\pi G\theta}{L} \int_0^R r^3 \, dr$$

$$= \frac{2\pi G\theta}{L} . \left[\frac{R^4}{4}\right]_0^R$$

$$= \frac{G\theta}{L} . \frac{2\pi R^4}{4}$$

$$= \frac{G\theta}{L} . \frac{\pi R^4}{2}$$

$= \frac{G\theta}{L} . \left[\frac{\pi d^4}{32}\right]$ now substituting $R = d/2$

$$= \frac{G\theta}{L} . J$$

since $\frac{\pi d^4}{32} = J$ the polar moment of inertia

or $\frac{T}{J} = \frac{G\theta}{L}$ ......(2)

if we combine the equation no.(1) and (2) we get $\boxed{\frac{T}{J} = \frac{\tau'}{r} = \frac{G.\theta}{L}}$

Where

T = applied external Torque, which is constant over Length L;

J = Polar moment of Inertia

$= \frac{\pi d^4}{32}$ for solid shaft

$= \frac{\pi (D^4 - d^4)}{32}$ for a hollow shaft. [D = Outside diameter ; d = inside diameter]

G = Modules of rigidity (or Modulus of elasticity in shear)

q = It is the angle of twist in radians on a length L.

Tensional Stiffness: The tensional stiffness k is defined as the torque per radius twist

Distribution of shear stresses in circular Shafts subjected to torsion :

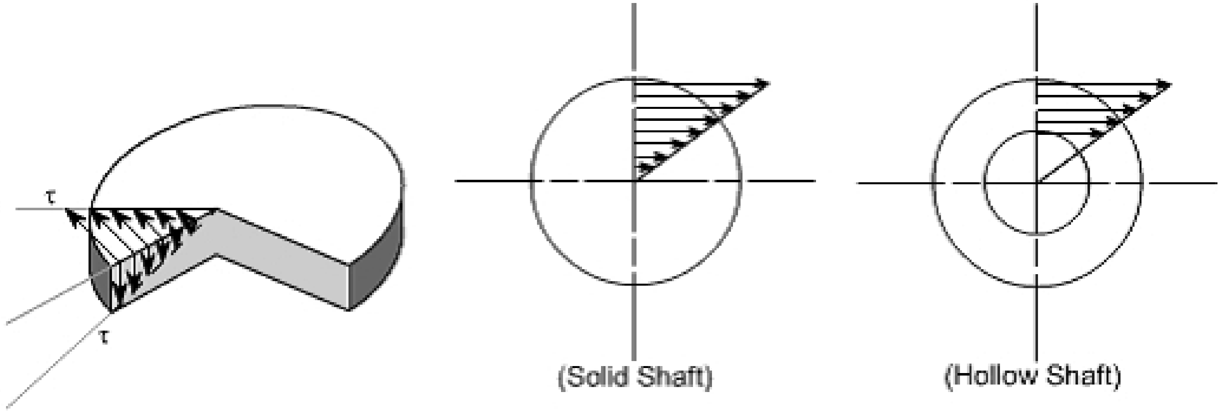
The simple torsion equation is written as

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

or

$$\tau = \frac{G\theta.r}{L}$$

This states that the shearing stress varies directly as the distance 'r' from the axis of the shaft and the following is the stress distribution in the plane of cross section and also the complementary shearing stresses in an axial plane.



Hence the maximum strear stress occurs on the outer surface of the shaft where r = R

The value of maximum shearing stress in the solid circular shaft can be determined as

$$\frac{\tau}{r} = \frac{T}{J}$$

$$\tau_{max}\Big|_{r=d/2} = \frac{T.R}{J} = \frac{T}{\frac{\pi d^4}{32}} . d/2$$

where d=diameter of solid shaft

$$\text{or} \quad \tau_{max^m} = \frac{16T}{\pi d^3}$$

From the above relation, following conclusion can be drawn

(i) t_{max}^{m} µ T

(ii) t_{max}^{m} µ $1/d^{3}$

Power Transmitted by a shaft:

In practical application, the diameter of the shaft must sometimes be calculated from the power which it is required to transmit.

Given the power required to be transmitted, speed in rpm 'N' Torque T, the formula connecting

These quantities can be derived as follows

$$P = T.\omega$$

$$= \frac{T.2\pi N}{60} \text{watts}$$

$$= \frac{2\pi NT}{60 \times 10^3} (\text{kw})$$

Torsional stiffness: The torsional stiffness k is defined as the torque per radian twist .

TORSION OF HOLLOW SHAFTS:

From the torsion of solid shafts of circular x – section , it is seen that only the material atthe outer surface of the shaft can be stressed to the limit assigned as an allowable working stresses. All of the material within the shaft will work at a lower stress and is not being used to full capacity. Thus, in these cases where the weight reduction is important, it is advantageous to use hollow shafts. In discussing the torsion of hollow shafts the same assumptions will be made as in the case of a solid shaft. The general torsion equation as we have applied in the case of torsion of solid shaft will hold good

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

For the hollow shaft

$$J = \frac{\pi(D_0^4 - d_i^4)}{32} \quad \text{where } D_0 = \text{Outside diameter}$$

$$d_i = \text{Inside diameter}$$

$$\text{Let } d_i = \frac{1}{2}.D_0$$

$$\tau_{max^m}\Big|_{solid} = \frac{16T}{\pi D_0^3} \qquad (1)$$

$$\tau_{max^m}\Big|_{hollow} = \frac{T.D_0/2}{\frac{\pi}{32}(D_0^4 - d_i^4)}$$

$$= \frac{16T.D_0}{\pi D_0^4\left[1-(d_i/D_0)^4\right]}$$

$$= \frac{16T}{\pi D_0^3\left[1-(1/2)^4\right]} = 1.066.\frac{16T}{\pi D_0^3} \qquad (2)$$

Closed Coiled helical springs subjected to axial loads:

Definition: A spring may be defined as an elastic member whose primary function is to deflect or distort under the action of applied load; it recovers its original shape when load is released.

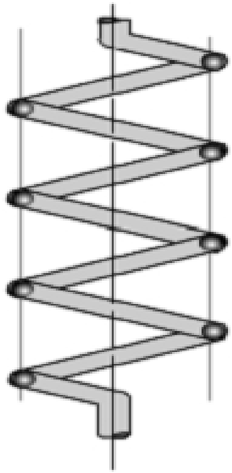
or

Springs are energy absorbing units whose function is to store energy and to restore it slowly or rapidly depending on the particular application.

Important types of springs are:

There are various types of springs such as

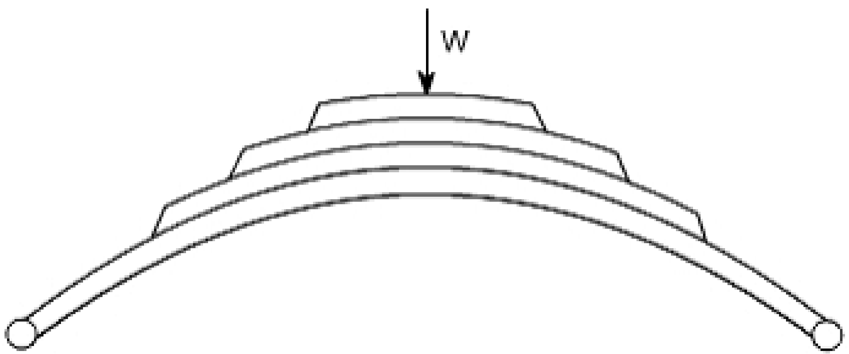
(i) helical spring: They are made of wire coiled into a helical form, the load being applied along the axis of the helix. In these type of springs the major stresses is torsional shear stress due to twisting. They are both used in tension and compression.



(ii) Spiral springs: They are made of flat strip of metal wound in the form of spiral and loaded in torsion.

In this the major stresses are tensile and compression due to bending.

(iv) Leaf springs: They are composed of flat bars of varying lengths clamped together so as to obtain greater efficiency . Leaf springs may be full elliptic, semi elliptic or cantilever types, In these type of springs the major stresses which come into picture are tensile & compressive.



These type of springs are used in the automobile suspension system.

Uses of springs :

(a) To apply forces and to control motions as in brakes and clutches.

(b) To measure forces as in spring balance.

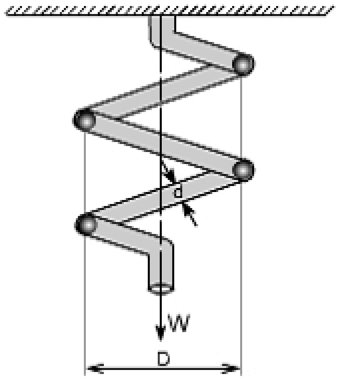
(c) To store energy as in clock springs.

(d) To reduce the effect of shock or impact loading as in carriage springs.

(e) To change the vibrating characteristics of a member as inflexible mounting of motors.

Derivation of the Formula :

In order to derive a necessary formula which governs the behaviour of springs, consider a closed coiled spring subjected to an axial load W.



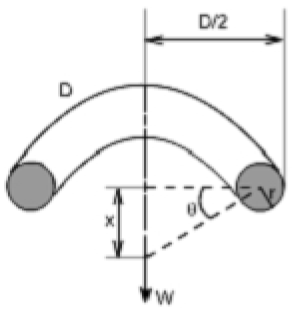
Let
W = axial load
D = mean coil diameter
d = diameter of spring wire
n = number of active coils
C = spring index = D / d For circular wires
l = length of spring wire
G = modulus of rigidity
x = deflection of spring
q = Angle of twist

when the spring is being subjected to an axial load to the wire of the spring gets be twisted like a shaft.

If q is the total angle of twist along the wire and x is the deflection of spring under the
action of load W along the axis of the coil, so that

x = D / 2 . q

again l = p D n [consider ,one half turn of a close coiled helical spring]



Using the torsion formula i.e

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

and substitituting $J = \frac{\pi d^4}{32}; T = w.\frac{d}{2}$

$$\theta = \frac{2.x}{D}; l = \pi D.x$$

Assumptions: (1) The Bending & shear effects may be neglected

(2) For the purpose of derivation of formula, the helix angle is considered to be
so small that it may be neglected.

Any one coil of a such a spring will be assumed to lie in a plane which is nearly ^r
to the
axis of the spring. This requires that adjoining coils be close together. With this limitation, a section taken perpendicular to the axis the spring rod becomes nearly
vertical. Hence to maintain equilibrium of a segment of the spring, only a shearing force
V = F and Torque T = F. r are required at any X – section. In the analysis of springs it is
customary to assume that the shearing stresses caused by the direct shear force is
uniformly distributed and is negligible
so applying the torsion formula.

SPRING DEFLECTION

$$\frac{w.d/2}{\frac{\pi d^4}{32}} = \frac{G.2x/D}{\pi D.n}$$

Thus,

$$x = \frac{8w.D^3.n}{G.d^4}$$

Spring striffness: The stiffness is defined as the load per unit deflection therefore

$$k = \frac{w}{x} = \frac{w}{\frac{8w.D^3.n}{G.d^4}}$$

Therefore

$$k = \frac{G.d^4}{8.D^3.n}$$

Shear stress

$$\frac{w.d/2}{\frac{\pi d^4}{32}} = \frac{\tau_{max^m}}{d/2}$$

$$\text{or } \tau_{max^m} = \frac{8wD}{\pi d^3}$$

WAHL'S FACTOR :

In order to take into account the effect of direct shear and change in coil curvature a stress factor is defined, which is known as Wahl's factor

K = Wahl' s factor and is defined as $K = \frac{4c-1}{4c-4} + \frac{0.615}{c}$

Where C = spring index

= D/d

if we take into account the Wahl's factor than the formula for the shear stress becomes

$$\tau_{max^m} = \frac{16.T.k}{\pi d^3}$$

Strain Energy : The strain energy is defined as the energy which is stored within a material when the work has been done on the material.

In the case of a spring the strain energy would be due to bending and the strain energy due to bending is given by the expansion

$$U = \frac{T^2 L}{2EI}$$

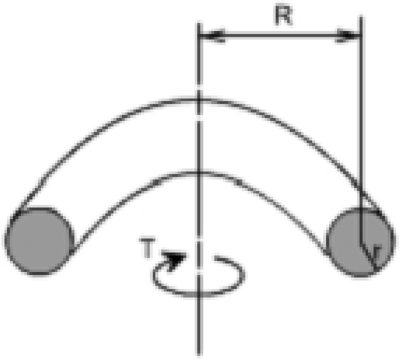
$$L = \pi D n$$

$$I = \frac{\pi d^4}{64}$$

so after substitution we get

$$U = \frac{32 T^2 D n}{E.d^4}$$

Close – coiled helical spring subjected to axial torque T or axial couple.



In this case the material of the spring is subjected to pure bending which tends to reduce Radius R of the coils. In this case the bending moment is constant through out the spring and is equal to the applied axial Torque T. The stresses i.e. maximum bending stress may thus be determined from the bending theory.

$$\sigma_{max} = \frac{M.y}{I}$$

$$= \frac{T.d/2}{\frac{\pi d^4}{64}}$$

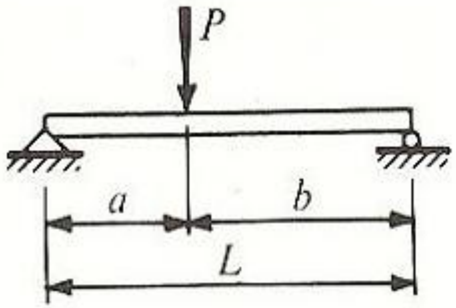
$$\sigma_{max} = \frac{32T}{\pi d^3}$$

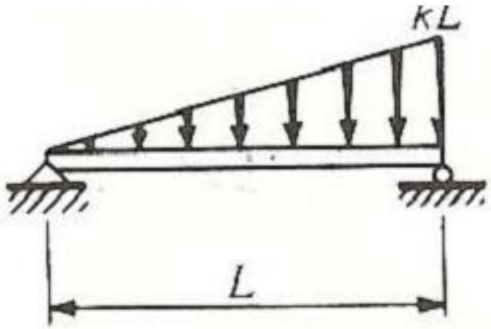
UNIT 3

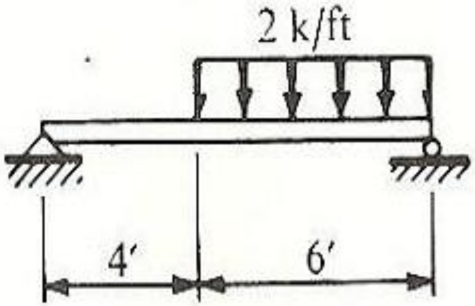
BEAMS

4-Classification of Beams:

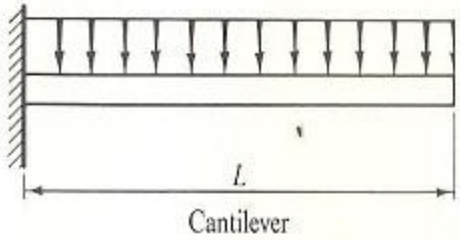
1) Simple Beam

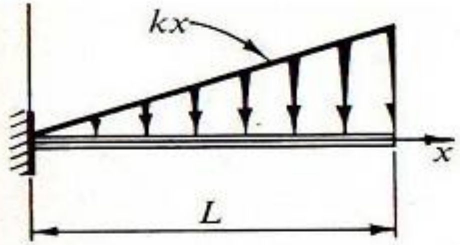


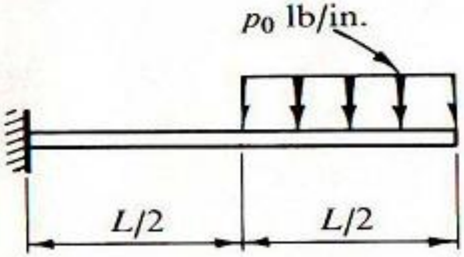




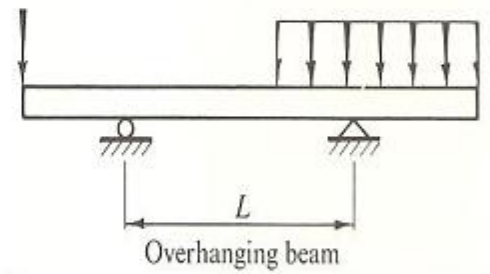
Cantilever Beam

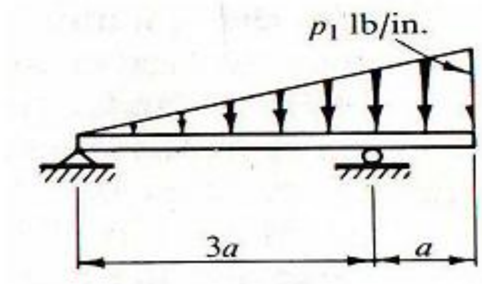






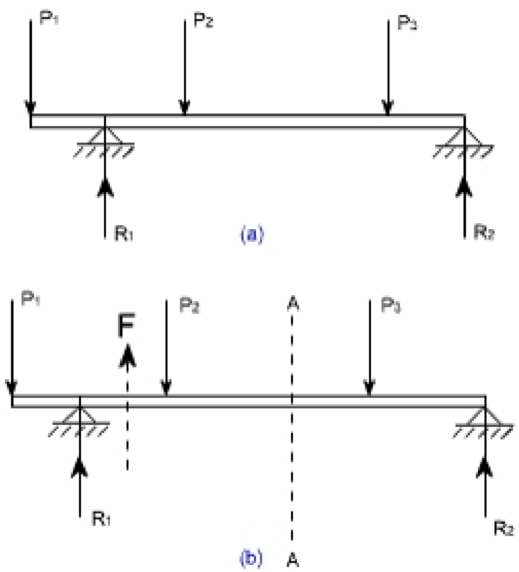
3) Simple Beam with Overhanging OR "Overhanging Beam"





Concept of Shear Force and Bending moment in beams:

When the beam is loaded in some arbitrarily manner, the internal forces and moments are developed and the terms shear force and bending moments come into pictures which are helpful to analyze the beams further. Let us define these terms



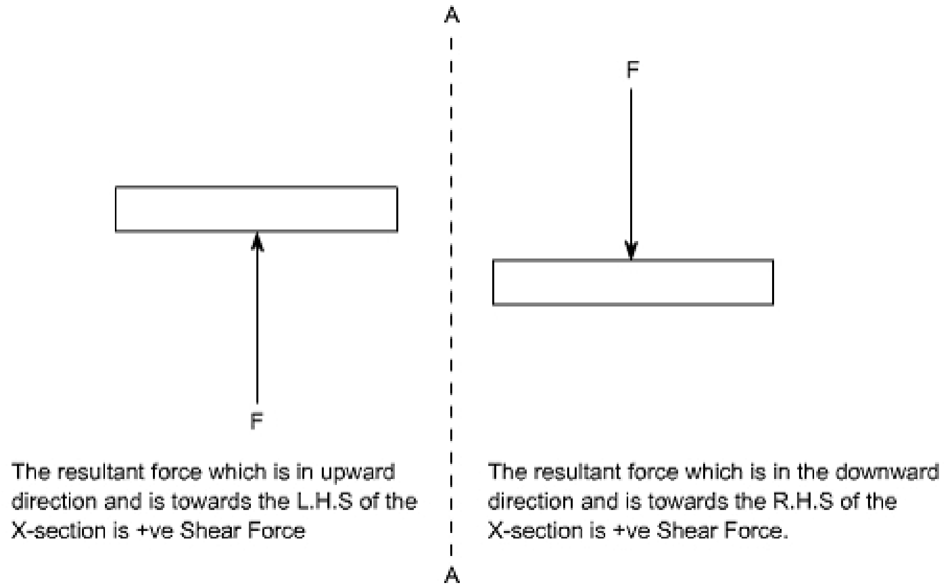
Now let us consider the beam as shown in fig 1(a) which is supporting the loads P_1, P_2, P_3 and is simply supported at two points creating the reactions R_1 and R_2 respectively. Now let us assume that the beam is to divided into or imagined to be cut into two portions at a section AA. Now let us assume that the resultant of loads and reactions to the left of AA is 'F' vertically upwards, and since the entire beam is to remain in equilibrium, thus the resultant of forces to the right of AA must also be F, acting downwards. This forces 'F' is as a shear force. The shearing force at any x-section of a beam represents the tendency for the portion of the beam to one side of the section to slide or shear laterally relative to the other portion.

Therefore, now we are in a position to define the shear force 'F' to as follows:

At any x-section of a beam, the shear force 'F' is the algebraic sum of all the lateral components of the forces acting on either side of the x-section.

Sign Convention for Shear Force:

The usual sign conventions to be followed for the shear forces have been illustrated in figures 2 and 3.



Positive Shear Force

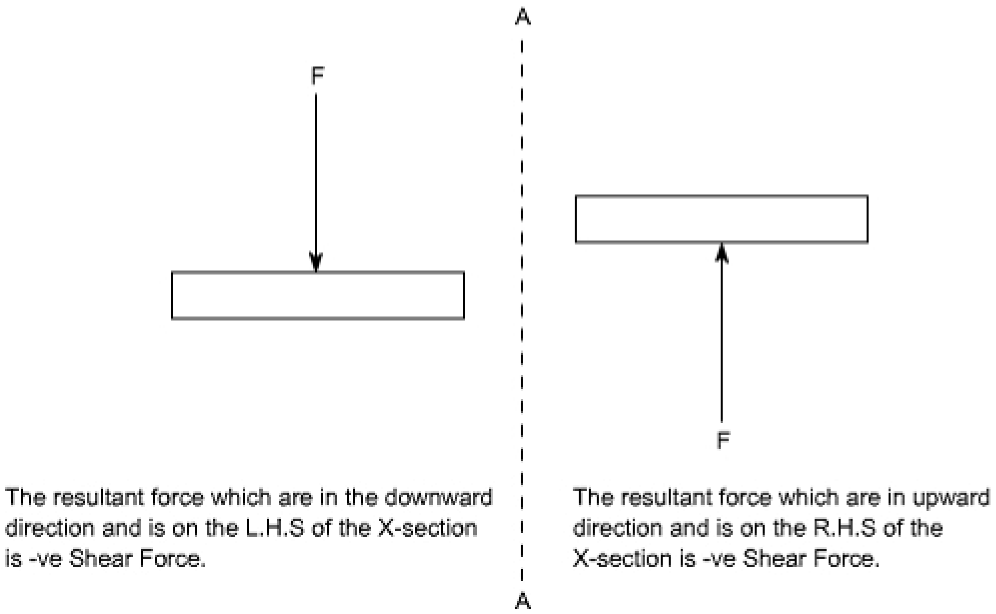
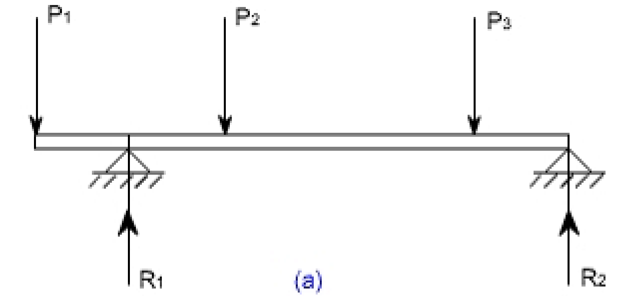
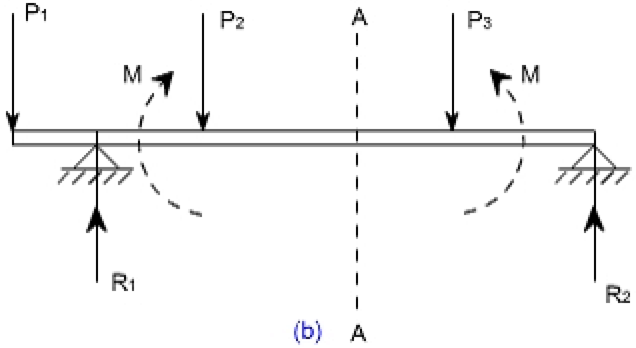


Fig 3: Negative Shear Force

BENDING MOMENT





Let us again consider the beam which is simply supported at the two prints, carrying loads P_1, P_2 and P_3 and having the reactions R_1 and R_2 at the supports Fig 4. Now, let us imagine that the beam is cut into two potions at the x-section AA. In a similar manner, as done for the case of shear force, if we say that the resultant moment about the section AA of all the loads and reactions to the left of the x-section at AA is M in C.W direction, then moment of forces to the right of x-section AA must be 'M' in C.C.W. Then 'M' is called as the Bending moment and is abbreviated as B.M. Now one can define the bending moment to be simply as <u>the algebraic sum of the moments about an x-section of all the forces acting on either side of the section</u>

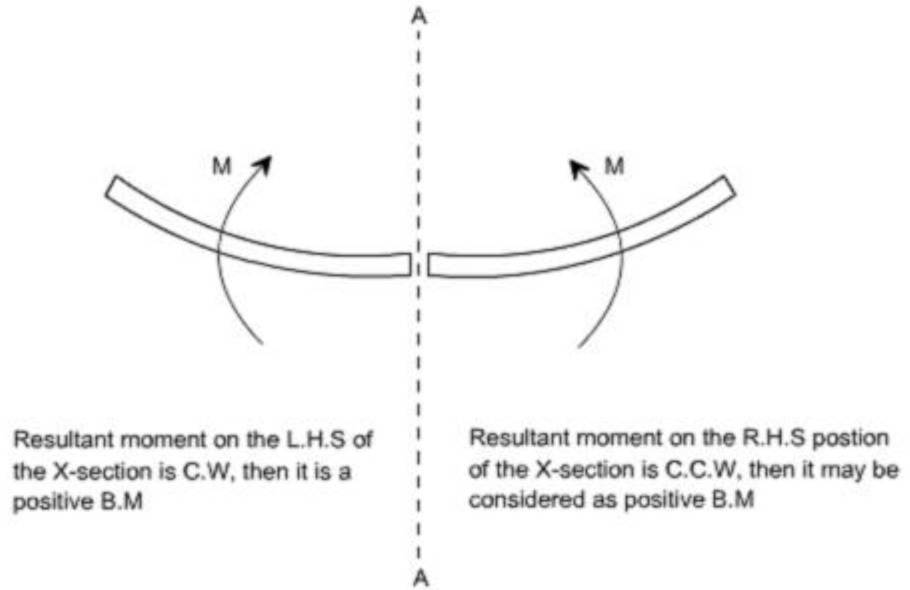
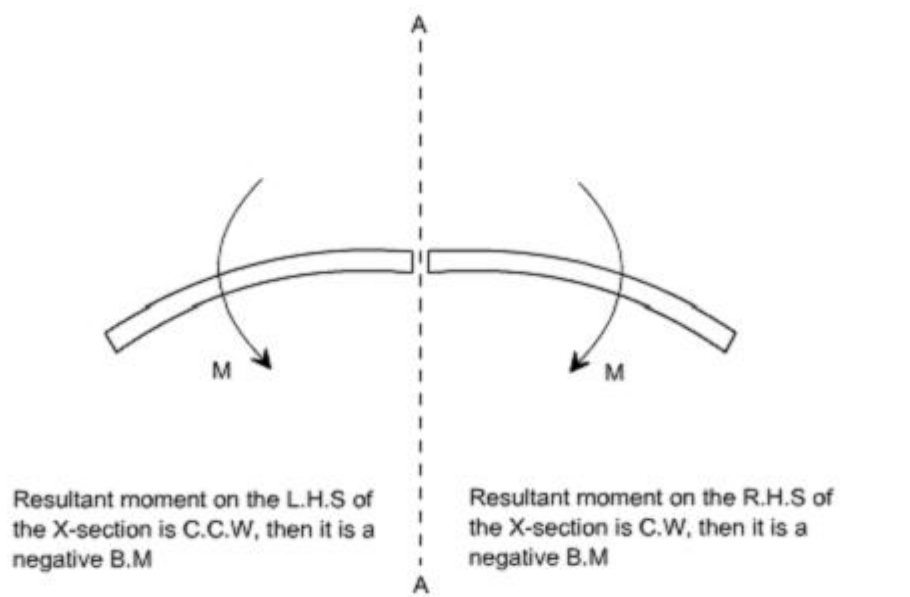
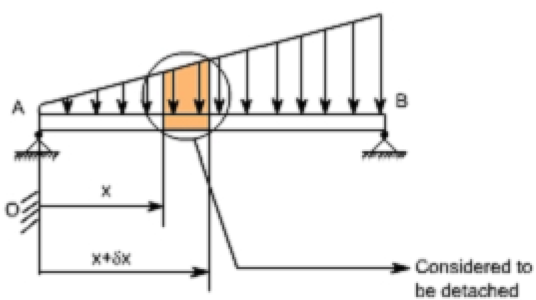


Fig 5: Positive Bending Moment

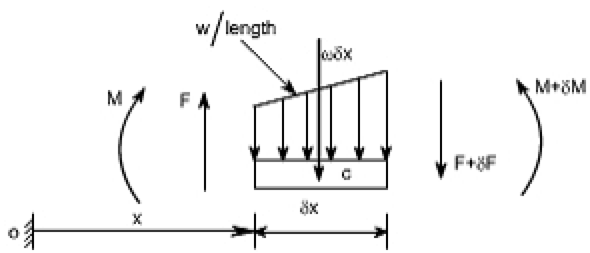


Basic Relationship Between The Rate of Loading, Shear Force and Bending Moment:

The construction of the shear force diagram and bending moment diagrams is greatly simplified if the relationship among load, shear force and bending moment is established. Let us consider a simply supported beam AB carrying a uniformly distributed load w/length. Let us imagine to cut a short slice of length dx cut out from this loaded beam at distance ‘x' from the origin ‘0'.



Let us detach this portion of the beam and draw its free body diagram.



The forces acting on the free body diagram of the detached portion of this loaded beam are the following

- The shearing force F and F+ dF at the section x and x + dx respectively.

- The bending moment at the sections x and x + dx be M and M + dM respectively.
- Force due to external loading, if ‘w' is the mean rate of loading per unit length then the total loading on this slice of length dx is w. dx, which is approximately acting through the centre ‘c'. If the loading is assumed to be uniformly distributed then it would pass exactly through the centre ‘c'.

This small element must be in equilibrium under the action of these forces and couples. Now let us take the moments at the point ‘c'. Such that

$$M + F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = M + \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + F.\frac{\delta x}{2} + \delta F.\frac{\delta x}{2} = \delta M \text{ [Neglecting the product of } \delta F \text{ and } \delta x \text{ being small quantities]}$$

$$\Rightarrow F.\delta x = \delta M$$

$$\Rightarrow F = \frac{\delta M}{\delta x}$$

Under the limits $\delta x \to 0$

$$\boxed{F = \frac{dM}{dx}} \qquad \text{............} \quad (1)$$

Resolving the forces vertically we get

$$w.\delta x + (F + \delta F) = F$$

$$\Rightarrow w = -\frac{\delta F}{\delta x}$$

Under the limits $\delta x \to 0$

$$\Rightarrow w = -\frac{dF}{dx} \text{ or } -\frac{d}{dx}\left(\frac{dM}{dx}\right)$$

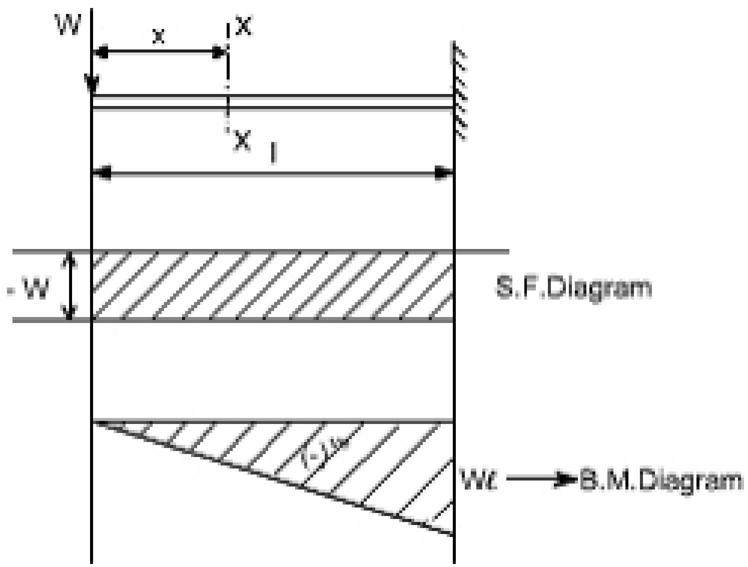
$$\boxed{w = -\frac{dF}{dx} = -\frac{d^2M}{dx^2}} \qquad \text{..........}(2)$$

A cantilever of length carries a concentrated load ‘W' at its free end.

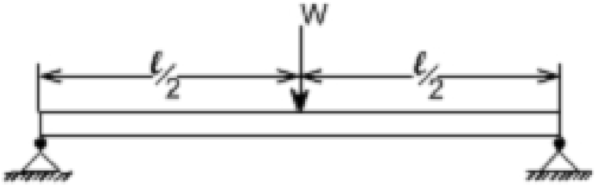
Draw shear force and bending moment.

Solution:

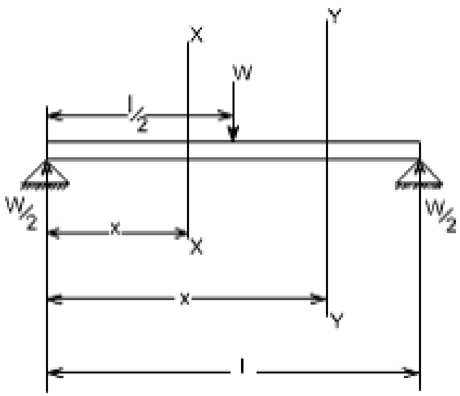
At a section a distance x from free end consider the forces to the left, then F = -W (for all values of x) -ve sign means the shear force to the left of the x-section are in downward direction and therefore negative. Taking moments about the section gives (obviously to the left of the section) M = -Wx (-ve sign means that the moment on the left hand side of the portion is in the anticlockwise direction and is therefore taken as –ve according to the sign convention) so that the maximum bending moment occurs at the fixed end i.e. M = -W l From equilibrium consideration, the fixing moment applied at the fixed end is Wl and the reaction is W. the shear force and bending moment are shown as,



Simply supported beam subjected to a central load (i.e. load acting at the mid-way)



By symmetry the reactions at the two supports would be W/2 and W/2. now consider any section X-X from the left end then, the beam is under the action of following forces.



.So the shear force at any X-section would be = W/2 [Which is constant upto x < l/2]

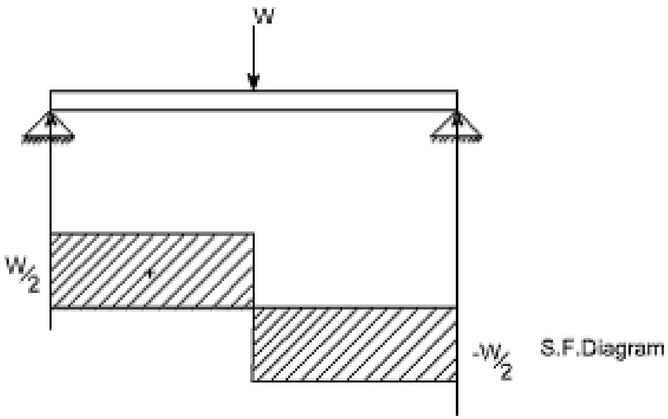
If we consider another section Y-Y which is beyond l/2 then

$$S.F_{Y\text{-}Y} = \frac{W}{2} - W = \frac{-W}{2}$$ for all values greater = l/2

Hence S.F diagram can be plotted as,

.For B.M diagram:

If we just take the moments to the left of the cross-section,



$$B.M_{x\text{-}x} = \frac{W}{2}x \text{ for } x \text{ lies between } 0 \text{ and } l/2$$

$$B.M_{\text{at } x = \frac{l}{2}} = \frac{W}{2}\,\frac{l}{2} \text{ i.e B.M at } x = 0$$

$$= \frac{Wl}{4}$$

$$B.M_{Y\text{-}Y} = \frac{W}{2}x - W\left(x - \frac{l}{2}\right)$$

Again

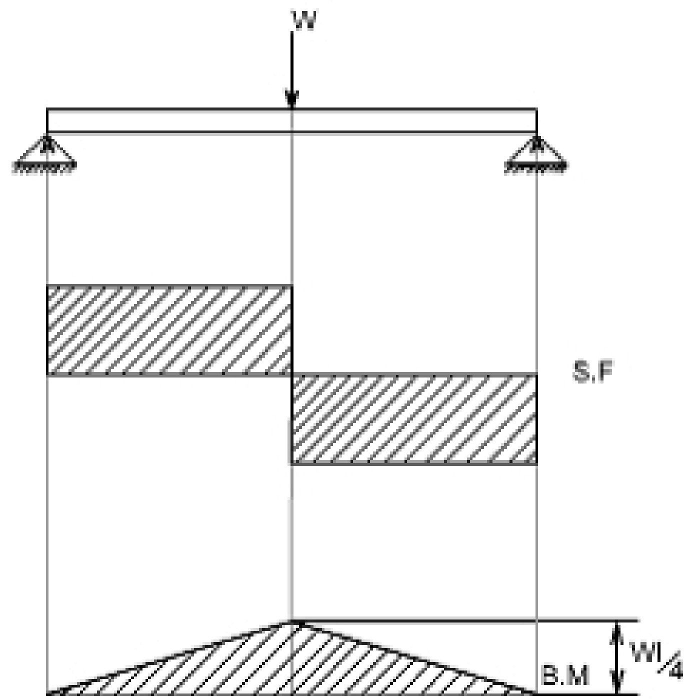
$$= \frac{W}{2}x - Wx + \frac{Wl}{2}$$

$$= -\frac{W}{2}x + \frac{Wl}{2}$$

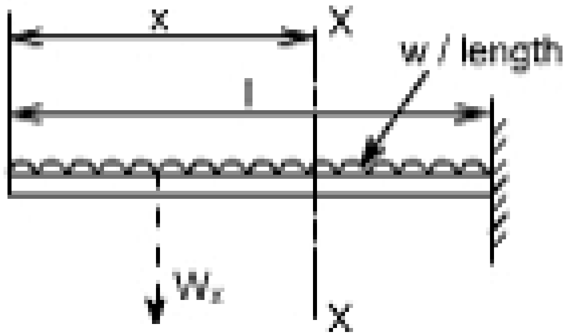
$$B.M_{\text{at } x = l} = -\frac{Wl}{2} + \frac{Wl}{2}$$

$$= 0$$

Which when plotted will give a straight relation i.e.



A cantilever beam subjected to U.d.L, draw S.F and B.M diagram.



Here the cantilever beam is subjected to a uniformly distributed load whose intensity is given w / length.

Consider any cross-section XX which is at a distance of x from the free end. If we just take the resultant of all the forces on the left of the X-section, then

$S.F_{xx} = -Wx$ for all values of 'x'. ---------- **(1)**

$S.F_{xx} = 0$

$S.F_{xx \text{ at } x=l} = -Wl$

So if we just plot the equation No. (1), then it will give a straight line relation. Bending Moment at X-X is obtained by treating the load to the left of X-X as a concentrated load of the same value acting through the centre of gravity.

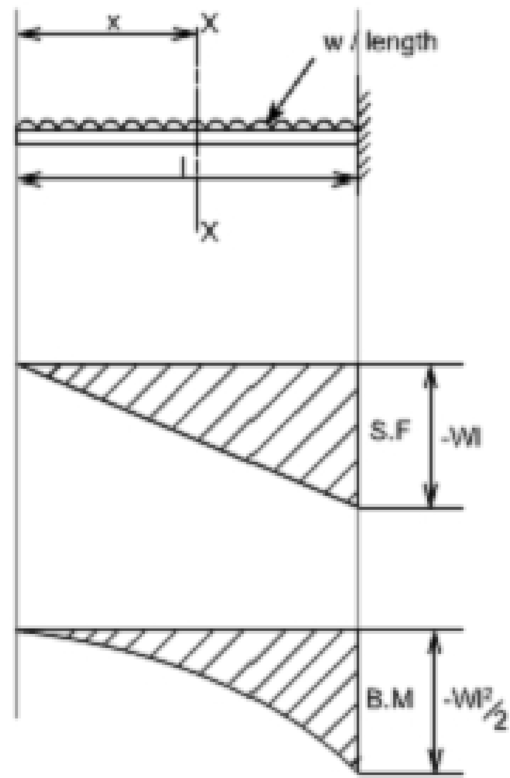
Therefore, the bending moment at any cross-section X-X is

$$B.M_{X\text{-}X} = -\,W\,x\,\frac{x}{2}$$
$$= -\,W\frac{x^2}{2}$$

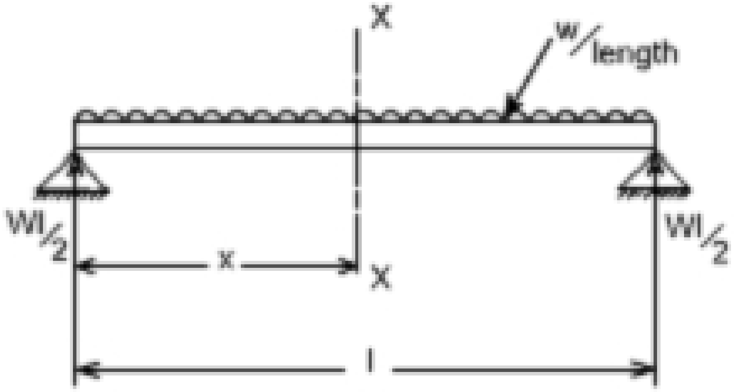
The above equation is a quadratic in x, when B.M is plotted against x this will produces a parabolic variation.

The extreme values of this would be at x = 0 and x = l

$$B.M_{at\ x\,=\,l} = -\frac{Wl^2}{2}$$
$$= \frac{Wl}{2} - Wx$$



Simply supported beam subjected to a uniformly distributed load U.D.L

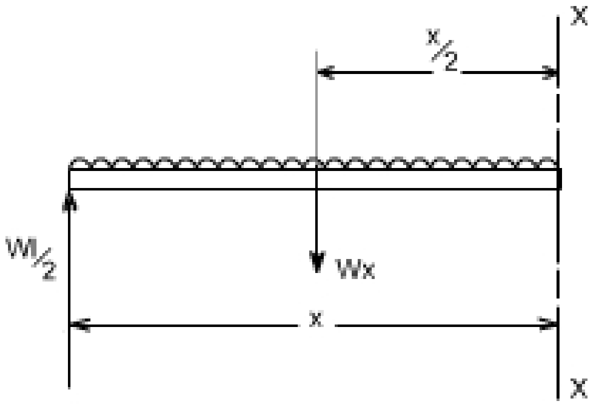


S.F at any X-section X-X is

$$= \frac{Wl}{2} - Wx$$

$$= W\left(\frac{l}{2} - x\right)$$

The bending moment at the section x is found by treating the distributed load as acting at its centre of gravity, which at a distance of x/2 from the section



$$B.M_{x-x} = \frac{Wl}{2}x - Wx.\frac{x}{2}$$

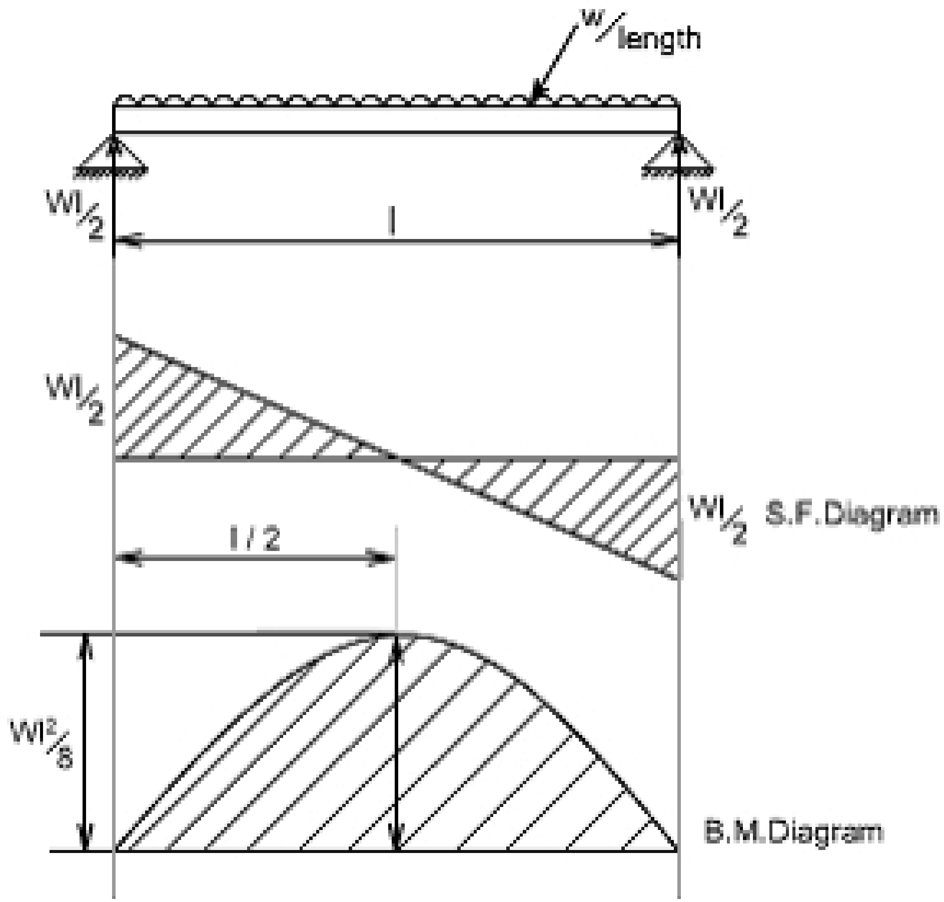
so the

$$= W.\frac{x}{2}(l - 2) \quad ....(2)$$

$$B.M_{at\ x=0} = 0$$

$$B.M_{at\ x=l} = 0$$

$$B.M\Big|_{at\ x=l} = -\frac{Wl^2}{8}$$



An I - section girder, 200mm wide by 300 mm depth flange and web of thickness is 20 mm is used as simply supported beam for a span of 7 m. The girder carries a distributed load of 5 KN /m and a concentrated load of 20 KN at mid-span.

Determine the

(i). The second moment of area of the cross-section of the girder

(ii). The maximum stress set up.

Solution:

The second moment of area of the cross-section can be determained as follows :

For sections with symmetry about the neutral axis, use can be made of standard I value for a rectangle about an axis through centroid i.e. (bd 3)/12. The section can thus be divided into convenient rectangles for each of which the neutral axis passes through the centroid. Example in the case enclosing the girder by a rectangle

$$I_{girder} = I_{rectangle} - I_{shaded\ portion}$$

$$= \left[\frac{200 \times 300^3}{12}\right] 10^{-12} - 2\left[\frac{90 \times 260^3}{12}\right] 10^{-12}$$

$$= (4.5 - 2.64)10^{-4}$$

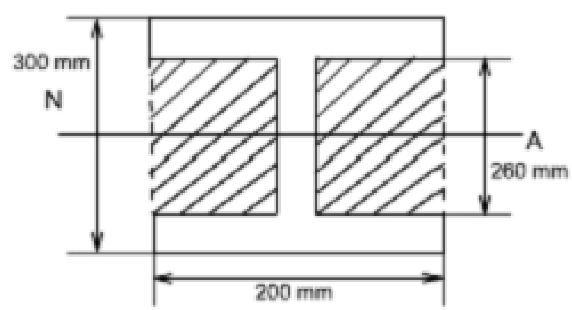
$$= 1.86 \times 10^{-4}\ m^4$$

The maximum stress may be found from the simple bending theory by equation

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

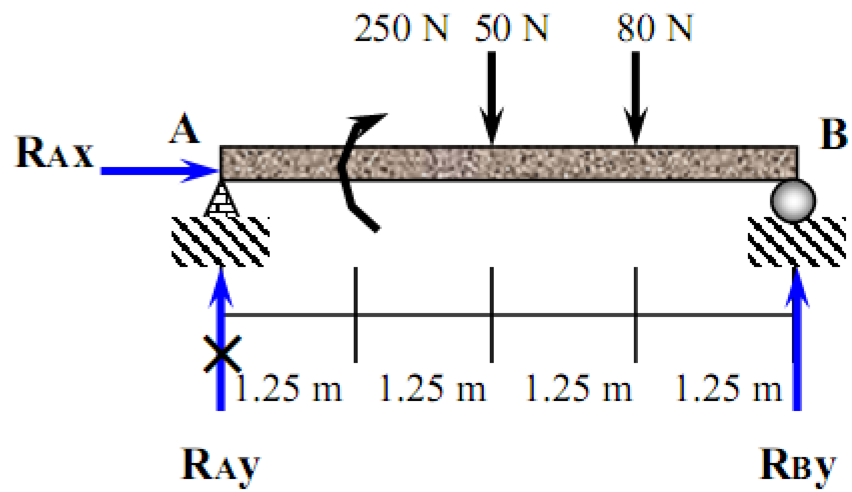
i.e.

$$\sigma_{max^m} = \frac{M_{max^m}}{I}\, y_{max^m}$$



Calculations of Beam Reactions

Ex3:



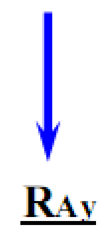
$\rightarrow \sum Fx = 0$ --- (1)

$R_{Ax} = 0$

$\circlearrowleft + \sum M@A = 0$ --- (2)

$250 + 80 \times 2.5 + 80 \times 3.75 - R_B \times 5 = 0$

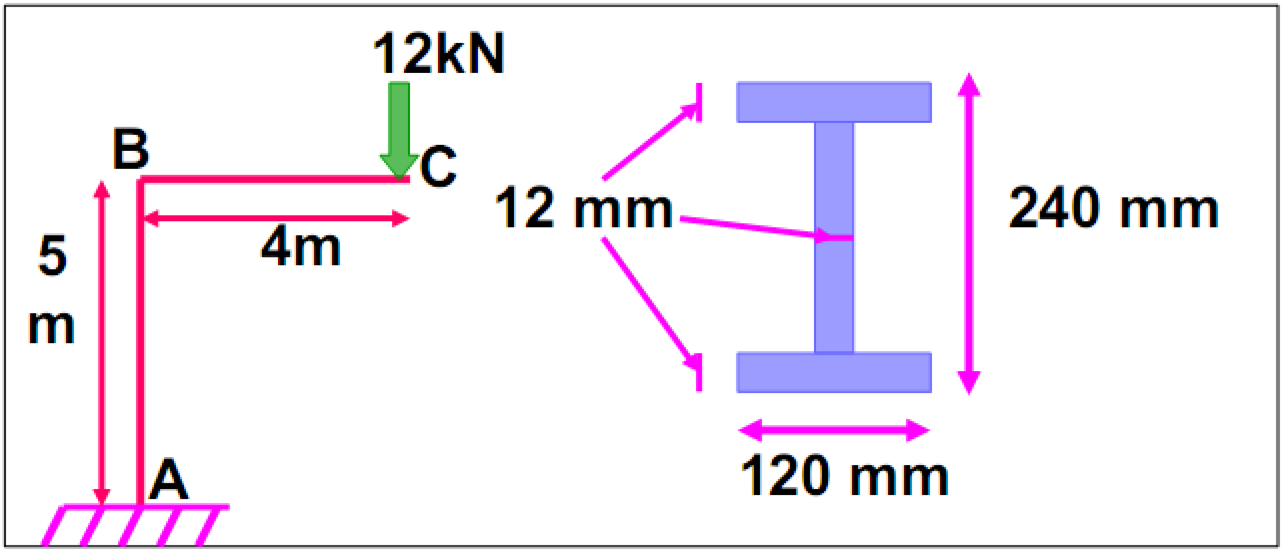
$\therefore R_{By} = +135\ N \uparrow$



$\uparrow \sum Fy = 0$ --- (3)

$R_{Ay} = -5\ N \uparrow \Rightarrow R_{Ay} = 5\ N \downarrow$

2. Compare the strain energies due to three types of internal forces in the rectangular bent shown in Fig. having uniform cross section shown in the same Fig. Take E=2 x 105 MPa, G= 0.8 x 105 MPa, Ar= 2736 mm^2



Solution:

Step 1: Properties

A=120 * 240 – 108 * 216 = 5472 mm2, $I = \frac{120*240^3}{12} - \frac{108*216^3}{12} = 47.54 \times 10^6\ mm^4$

E= 2 * 105 MPa ; G= 0.8 * 105 MPa ; Ar = 2736 mm^2

Step 2: Strain Energy due to Axial Forces

Member AB is subjected to an axial comprn.=-12 kN

Strain Energy due to axial load for the whole str. is

$$(U_i)_P = \sum_{i=1}^{n=2} \frac{P^2 L}{2AE} = \frac{(-12*10^3)^2 * 5000}{2*5472*2*10^5} = 328.94\ N\text{-}mm$$

Step 3: Strain Energy due to Shear Forces

Shear force in AB = 0; Shear force in BC = 12 kN

Strain Energy due to Shear for the whole str. Is

$$(U_i)_V = \sum_{i=1}^{n=2} \frac{V_x^2 L}{2A_r G} = \frac{(12*10^3)^2 * 4000}{2*2736*0.8\,x10^5} = 1315.78\ \text{N - mm}$$

Step 4: **Strain Energy due to Bending Moment**

Bending Moment in AB = -12 * 4 = -48 kN-m

Bending Moment in BC = -12 x

Strain Energy due to BM for the whole structure is

$$(U_i)_M = \sum_{i=1}^{n=2} \frac{M_x^2 dx}{2EI} = \frac{(-48*10^6)^2 * 5000}{2*2x10^5 * 47.54*10^6} + \int_0^{4000} \frac{(-12*10^3 * x)^2 dx}{2*2*10^5 * 47.54*10^6} = 767.34*10^3\ \text{N - mm}$$

Step 5: **Comparison**

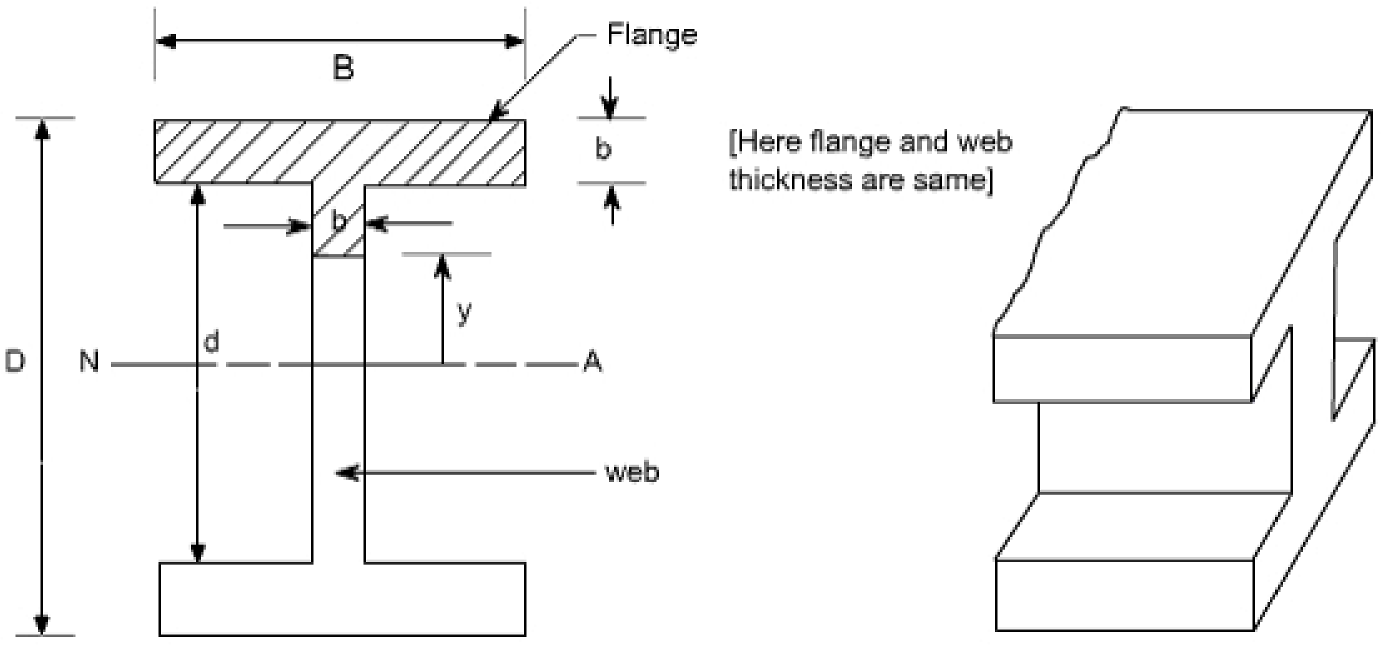
Total Strain Energy = (Ui)p + (Ui)V+ (Ui)M

Total Strain Energy =328.94 +1315.78 +767.34 x 103

= 768.98 x 103 N-mm

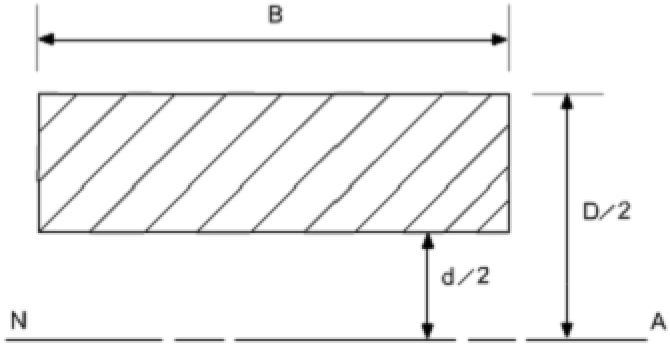
Strain Energy due to axial force, shear force and bending moment are 0.043%, 0.17% & 99.78 % of the total strain energy.

Consider an I - section of the dimension shown below.



The shear stress distribution for any arbitrary shape is given as $\tau = \frac{F A \bar{y}}{Z I}$

Let us evaluate the quantity $A\bar{y}$, the $A\bar{y}$ quantity for this case comprise the contribution due to flange area and web area



Flange area

Area of the flange $=B\left(\frac{D-d}{2}\right)$

Distance of the centroid of the flange from the N.A

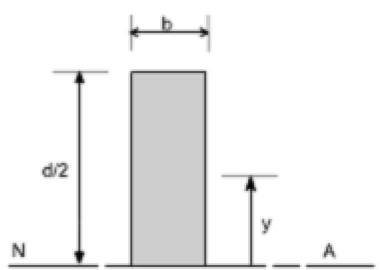
$$\bar{y}=\frac{1}{2}\left(\frac{D-d}{2}\right)+\frac{d}{2}$$

$$\bar{y}=\left(\frac{D+d}{4}\right)$$

Hence,

$$A\bar{y}\big|_{Flange}=B\left(\frac{D-d}{2}\right)\left(\frac{D-d}{4}\right)$$

Web Area



Area of the web

$$A=b\left(\frac{d}{2}-y\right)$$

Distance of the centroid from N.A

$$\bar{y}=\frac{1}{2}\left(\frac{d}{2}-y\right)+y$$

$$\bar{y}=\frac{1}{2}\left(\frac{d}{2}+y\right)$$

Therefore,

$$A\bar{y}\big|_{web}=b\left(\frac{d}{2}-y\right)\frac{1}{2}\left(\frac{d}{2}+y\right)$$

Hence,

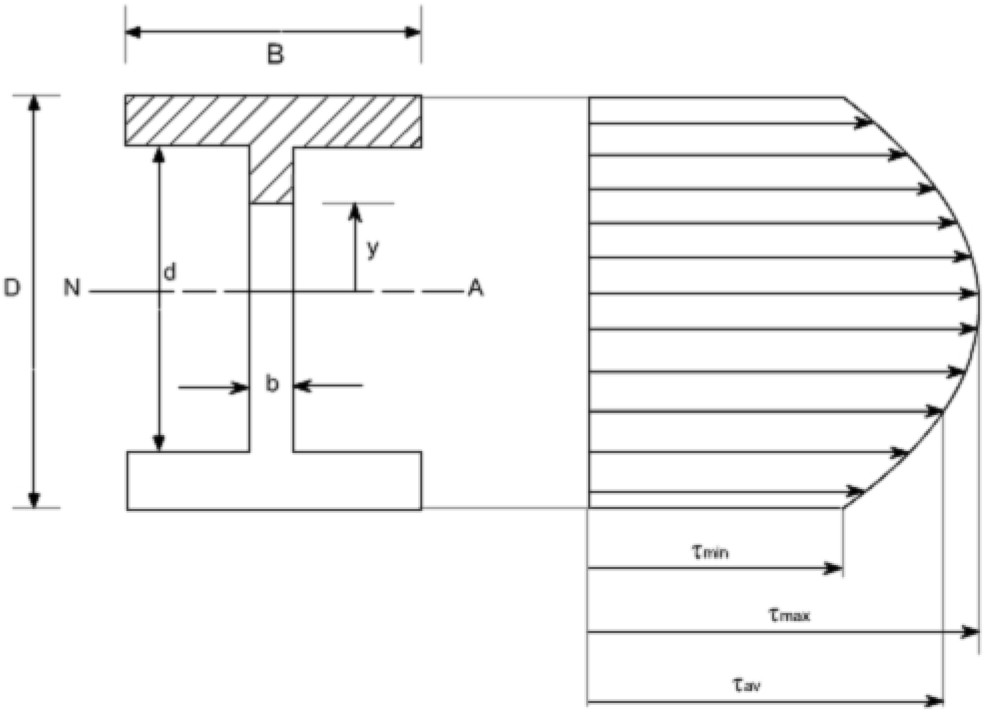
$$A\bar{y}\big|_{Total}=B\left(\frac{D-d}{2}\right)\left(\frac{D+d}{4}\right)+b\left(\frac{d}{2}-y\right)\left(\frac{d}{2}+y\right)\frac{1}{2}$$

Thus,

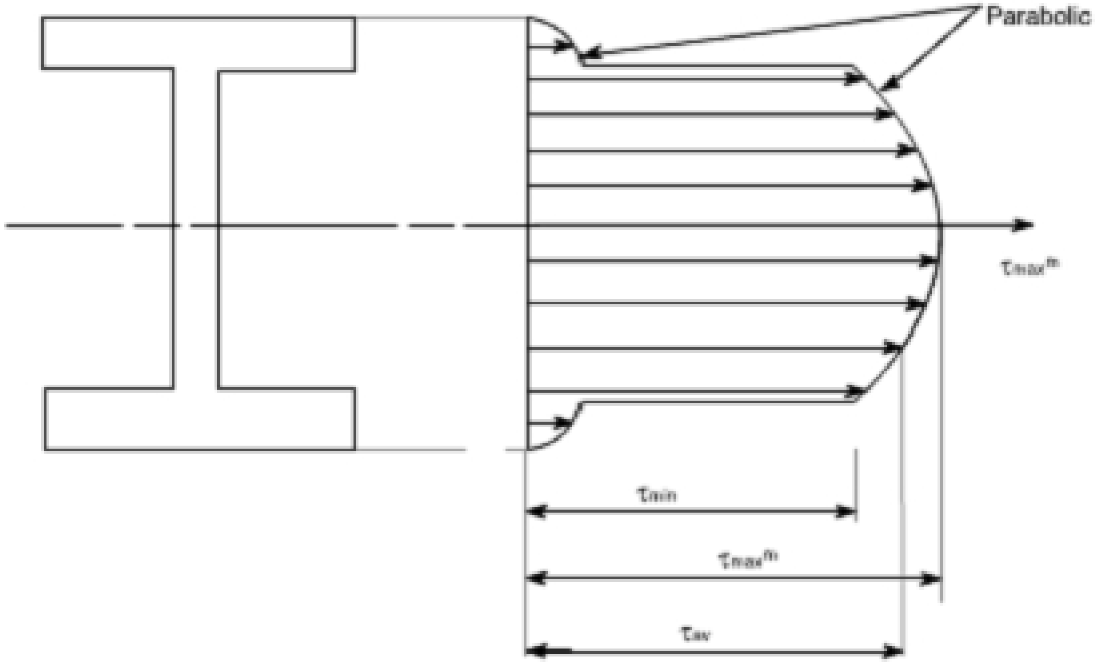
$$A\bar{y}\big|_{Total}=B\left(\frac{D^2-d^2}{8}\right)+\frac{b}{2}\left(\frac{d^2}{4}-y^2\right)$$

Therefore shear stress,

$$\tau=\frac{F}{bI}\left[\frac{B\left(D^2-d^2\right)}{8}+\frac{b}{2}\left(\frac{d^2}{4}-y^2\right)\right]$$



$$\tau_{max^m} = \frac{F}{8bI}\left[B\left(D^2 - d^2\right) + bd^2\right]$$



This distribution is known as the "top – hat" distribution. Clearly the web bears the most of the shear stress and bending theory we can say that the flange will bear most of the bending stress.

UNIT 4

DEFLECTION OF BEAMS

Deflection of Beams

The deformation of a beam is usually expressed in terms of its deflection from its original unloaded position. The deflection is measured from the original neutral surface of the beam to the neutral surface of the deformed beam. The configuration assumed by the deformed neutral surface is known as the elastic curve of the beam.

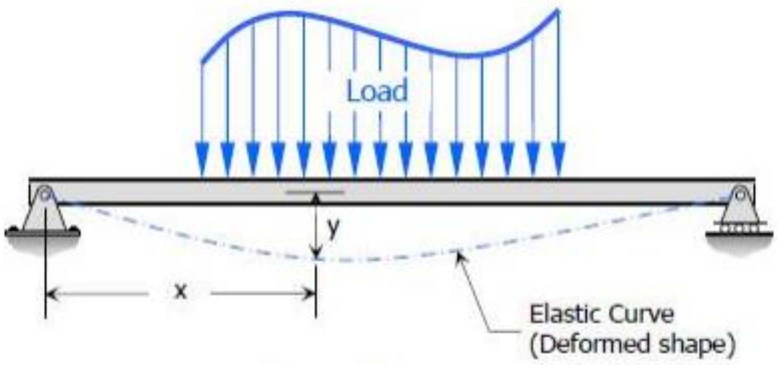


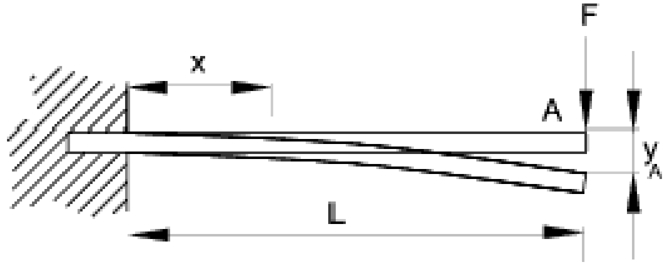
Figure: Elastic curve

METHODS OF DETERMINING DEFLECTION OF BEAMS

- Double integration method
- Moment area method
- Conjugate method
- Macaulay's method

Example - Cantilever beam

Consider a cantilever beam (uniform section) with a single concentrated load at the end. At the fixed end x = 0, dy = 0 , dy/dx = 0



From the equilibrium balance ..At the support there is a resisting moment - FL and a vertical upward force F.
At any point x along the beam there is a moment $F(x - L) = M_x = EI\, d^2y/dx^2$

$$EI \frac{d^2y}{dx^2} = -F(L-x) \quad \text{Integrating}$$

$$EI \frac{dy}{dx} = -F\left(Lx - \frac{x^2}{2}\right) + C_1 \; \text{.....}(C_1 = 0 \text{ because } dy/dx = 0 \text{ at } x = 0)$$

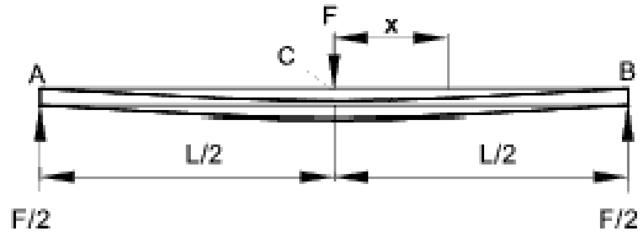
Integrating again

$$EI\, y = -F\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2 \; \text{.....}(C = 0 \text{ because } y = 0 \text{ at } x = 0)$$

$$\text{At end A} \quad \left(\frac{dy}{dx}\right)_A = -\frac{F}{EI}\left(L^2 - \frac{L^2}{2}\right) = -\frac{FL^2}{2EI} \quad \text{and} \quad y_A = -\frac{F}{EI}\left(\frac{L^3}{2} - \frac{L^3}{6}\right) = -\frac{FL^3}{3EI}$$

Example - Simply supported beam

Consider a simply supported uniform section beam with a single load F at the centre. The beam will be deflect symmetrically about the centre line with 0 slope (dy/dx) at the centre line. It is convenient to select the origin at the centre line.



$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{F}{2}\left(\frac{L}{2}+x\right) - Fx\right] = \frac{F}{2EI}\left(\frac{L}{2}-x\right)$$ Integrating

$$\frac{dy}{dx} = \frac{F}{2EI}\left(\frac{Lx}{2} - \frac{x^2}{2}\right) + C_1$$ .. ($C_1 = 0$ because $dy/dx = 0$ at $x = 0$)

Integrating again $y = \frac{F}{2EI}\left(\frac{Lx^2}{4} - \frac{x^3}{6}\right) + C_2$

$y = 0$ when $x = L/2$ therefore $\frac{F}{2EI}\left(\frac{L^3}{8} - \frac{L^3}{12}\right) + c_2 = 0$

and thus $c_2 = -\frac{FL^3}{48EI}$

At end B $\left(\frac{dy}{dx}\right)_b = -\frac{F}{2EI}\left(\frac{L^2}{4} - \frac{L^2}{8}\right) = \frac{FL^2}{16EI}$ and $y_B = \frac{F}{2EI}\left(\frac{L^3}{8} - \frac{L^3}{12}\right) - \frac{FL^3}{48EI} = 0$

At centre C $x = 0$ $y_c = -\frac{Fl^3}{48EI}$ (slope $\frac{dy}{dx} = 0$ by symmetry)

Moment Area Method

This is a method of determining the change in slope or the deflection between two points on a beam. It is expressed as two theorems...

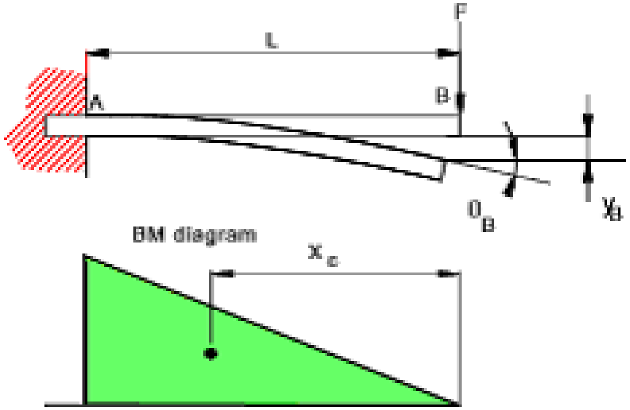
Theorem 1

If A and B are two points on a beam the change in angle (radians) between the tangent at A and the tangent at B is equal to the area of the bending moment diagram between the points divided by the relevant value of EI (the flexural rigidity constant).

Theorem 2

If A and B are two points on a beam the displacement of B relative to the tangent of the beam at A is equal to the moment of the area of the bending moment diagram between A and B about the ordinate through B divided by the relevant value of EI (the flexural rigidity constant).

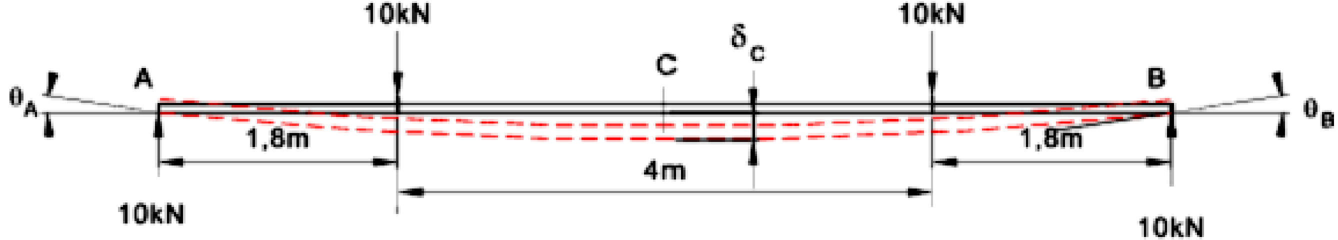
Examples ..Two simple examples are provide below to illustrate these theorems
Example 1) Determine the deflection and slope of a cantilever as shown..

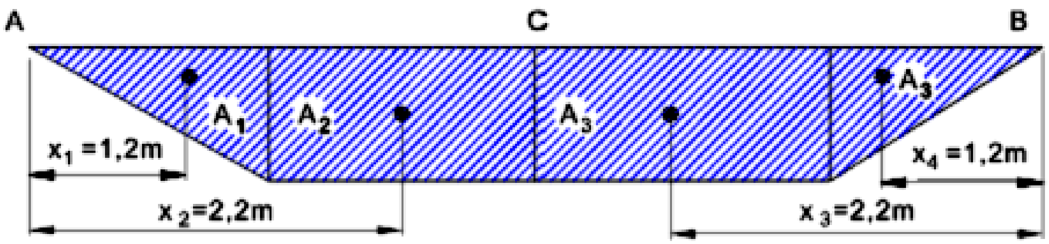


The bending moment at A = M_A = FL
The area of the bending moment diagram $A_M = F.L^2 /2$
The distance to the centroid of the BM diagram from B= $x_c = 2L/3$
The deflection of B = $y_b = A_M.x_c /EI = F.L^3 /3EI$
The slope at B relative to the tan at A = $\theta_b = A_M /EI = FL^2 /2EI$

Example 2) Determine the central deflection and end slopes of the simply supported beam as shown..

$E = 210$ GPa ......$I = 834$ cm^4...... $EI = 1{,}7514.\ 10^6 Nm^2$





Bending Moment Diagram

$A_1 = 10.1{,}8.1{,}8/2 = 16{,}2kNm$
$A_2 = 10.1{,}8.2 = 36kNm$
$A_2 = 10.1{,}8.2 = 36kNm$
$A_1 = 10.1{,}8.1{,}8/2 = 16{,}2kNm$
x_1 = Centroid of $A_1 = (2/3).1{,}8 = 1{,}2$
x_2 = Centroid of $A_2 = 1{,}8 + 1 = 2{,}8$
x_3 = Centroid of $A_3 = 1{,}8 + 1 = 2{,}8$
x_4 = Centroid of $A_4 = (2/3).1{,}8 = 1{,}2$

The slope at A is given by the area of the moment diagram between A and C divided by EI.

$$\theta_A = (A_1 + A_2)/EI = (16{,}2+36).10^3 / (1{,}7514.\ 10^6)$$
$$= 0{,}029\text{rads} = 1{,}7 \text{ degrees}$$

The deflection at the centre (C) is equal to the deviation of the point A above a line that is tangent to C.
Moments must therefore be taken about the deviation line at A.

$$\delta_C = (A_M.x_M)/EI = (A_1 x_1 + A_2 x_2)/EI = 120{,}24.10^3/(1{,}7514.\ 10^6)$$
$$= 0{,}0686\text{m} = 68{,}6\text{mm}$$

Moment Area Method

This method is based on two theorems which are stated through an example. Consider a beam AB subjected to some arbitrary load as shown in Figure 1.

Let the flexural rigidity of the beam be EI. Due to the load, there would be bending moment and BMD would be as shown in Figure 2. The deflected shape of the beam which is the elastic curve is shown in Figure 3. Let C and D be two points arbitrarily chosen on the beam. On the elastic curve, tangents are drawn at deflected positions of C and D. The angles made by these tangents with respect to the horizontal are marked as and . These angles are nothing but slopes. The change is the angle between these two tangents is demoted as . This change in the angel is equal to the area of the diagram between the two points C and D. This is the area of the shaded portion in figure 2.

Hence $\theta_{CD} = \theta_C \sim \theta_D$ = Area of $\frac{M}{EI}$ diagram between C and D

$$\theta_{CD} = \frac{\text{Area BM}}{EI} \longrightarrow 1\ (a)$$

It is also expressed in the integration mode as

$$\theta_{CD} = \int_{CD} \frac{M}{EI} dx \longrightarrow 1\ (b)$$

Equation 1 is the first moment area theorem which is stated as follows:

Statement of theorem I:

The change in slope between any two points on the elastic curve for a member subjected to bending is equal to the area of $\frac{M}{EI}$ ***diagram between those two points.***

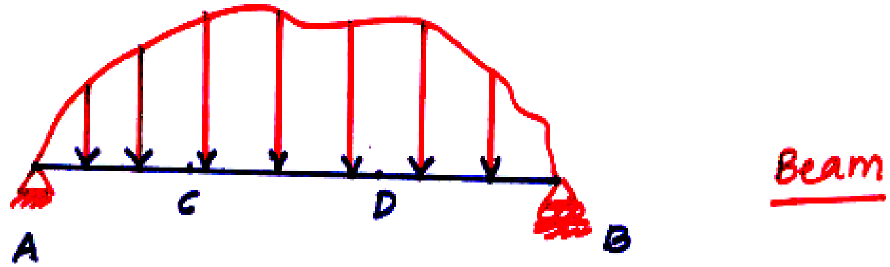


Fig. 1

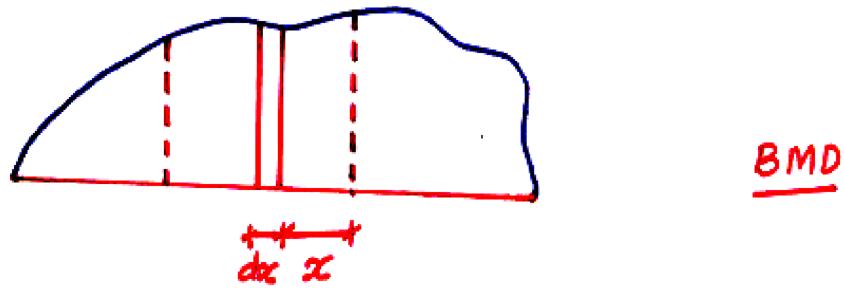


Fig. 2

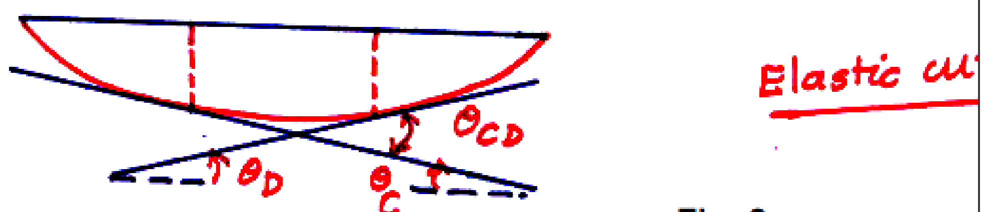


Fig. 3

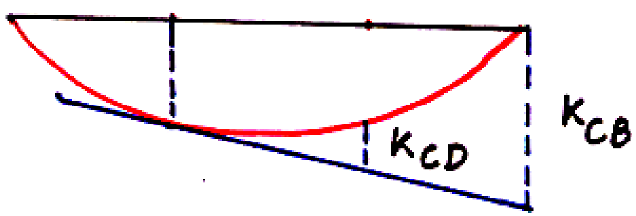
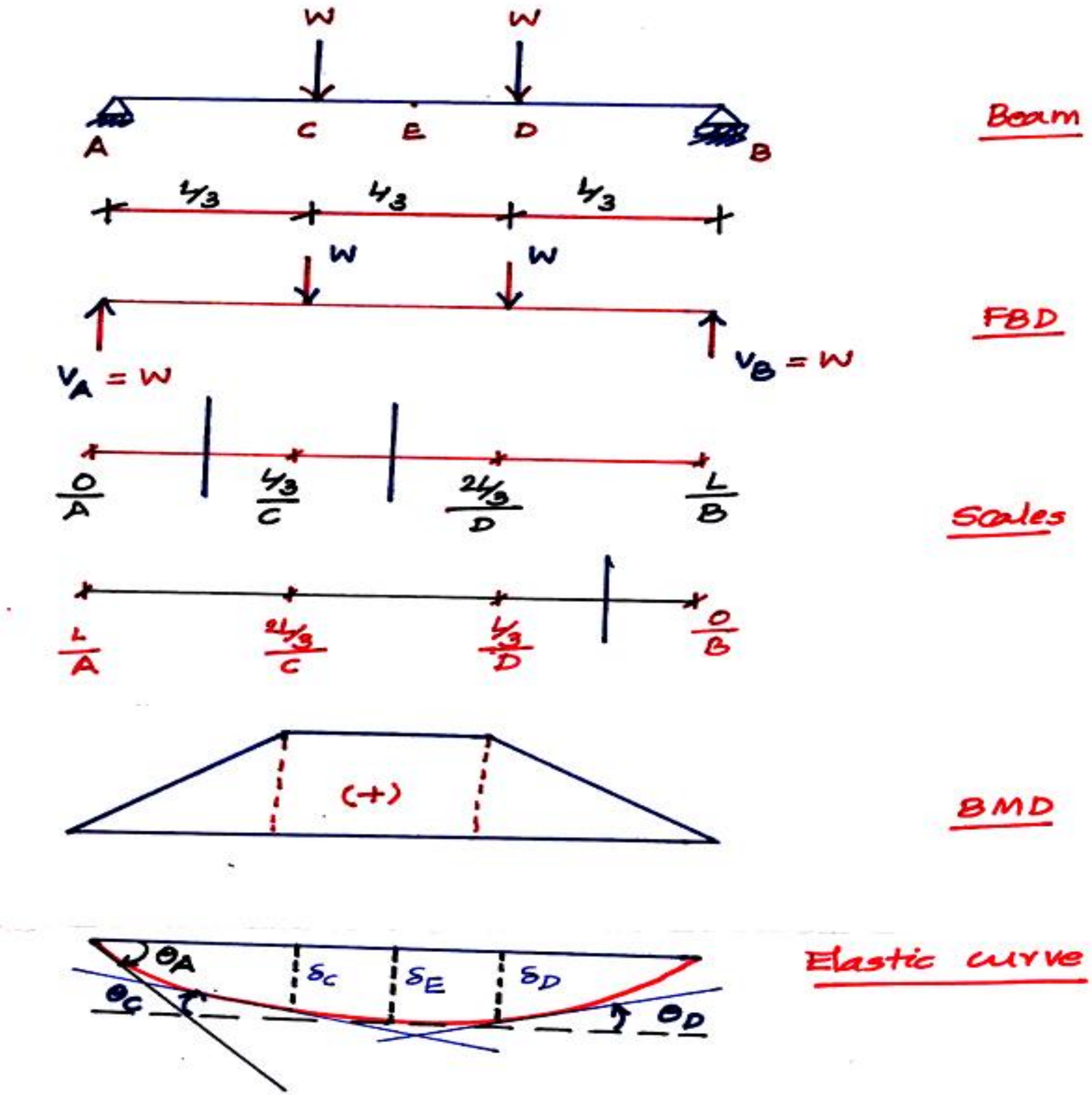


Fig. 4

Problem 1 : Compute deflections and slopes at C,D and E. Also compute slopes at A and B.



To Compute Reactions:

$$\overset{\rightarrow +}{\sum} fx = 0 \Rightarrow H_A = 0$$

$$\uparrow + \sum fy = 0 \Rightarrow V_A + V_B - W - W = 0$$

$$V_A + V_B = 2W$$

$$+\circlearrowright \sum M_B = 0 \Rightarrow LV_A - \frac{WL}{3} - W\frac{(2L)}{3} = 0$$

$$LV_A = \frac{WL}{3} + \frac{2WL}{3} = WL$$

$$V_A = W \quad ; \quad V_B = W$$

Bending Moment Calculations:

Section (1) – (1) (LHP, 0 to L/3)

$+\circlearrowright \quad M_{x-x} = Wx$

At x = 0; BM at A = 0

$x = L/3$; BM @ C = $WL/3$

Section (2) – (2) (LHP, $L/3$ to $2L/3$)

$+\circlearrowright \quad M_{x-x} = Wx - W(x - L/3)$

At $x = L/3$, BM @ C = $W L/3 - W L/3 + W L/3$

$= W L/3$

$$\text{At } x = \frac{2L}{3}, \text{ BM @ D} = W\left(\frac{2L}{3}\right) - W\left(\frac{2L}{3} - \frac{L}{3}\right)$$

$$= \left(\frac{2WL}{3} - \frac{2WL}{3} + \frac{WL}{3}\right)$$

$$= \frac{WL}{3}$$

Section (3) – (3) RHP (0 to $L/3$)

$\circlearrowleft + \quad M_{x-x} = Wx$

At x = 0; BM @ B = 0

$$\text{At } x = L/3, \text{ BM @ D} = \frac{WL}{3}$$

This beam is symmetrical. Hence the BMD & elastic curve is also symmetrical. In such a case, maximum deflection occurs at mid span, marked as δ_E. Thus, the tangent drawn at E will be parallel to the beam line and θ_E is zero.

Also, $\delta_c = \delta_D$, $\theta_A = \theta_B$ and $\theta_C = \theta_D$

To compute θ_C

From first theorem,

$$\theta_{CE} = \frac{\text{Area of BMD between E\&C}}{EI}$$

$$\theta_{C} - \theta_{E} = \frac{W\frac{L}{3}\left(\frac{L}{6}\right)}{EI}$$

$$= \frac{WL^2}{18EI}$$

θ_E being zero, $\theta_C = \frac{WL^2}{18EI}$ (↺)

To compute θ_A

From First theorem,

$$\theta_{AE} = \frac{\text{Area of BMD between A\&E}}{EI}$$

$$\theta_{A} - \theta_{E} = \frac{\frac{1}{2}\left(\frac{L}{3}\right)\frac{WL}{3} + \frac{WL}{3}\left(\frac{L}{6}\right)}{EI}$$

$$= \frac{\frac{WL^2}{18} + \frac{WL^2}{18}}{EI}$$

θ_E being zero, $\theta_A = \frac{WL^2}{9EI}$ (↻)

$$\theta_B = \frac{WL^2}{9EI} \quad (↺)$$

To compute δ_E

From 2nd theorem

$$K_{EA} = \frac{\left(\text{Area of BM } \bar{X}\right)_{EA}}{EI}$$

$$K_{EA} = \frac{\left(\frac{1}{2}\frac{L}{3}\frac{WL}{3}\right)\left(\frac{2}{3}\frac{L}{3}\right) + \left(\frac{WL}{3}\frac{L}{6}\right)\left(\frac{L}{3}+\frac{L}{12}\right)}{EI}$$

$$= \frac{\frac{WL^3}{81} + \frac{5WL^3}{216}}{EI}$$

$$= \frac{1}{EI}\left\{\frac{8WL^3 + 15WL^3}{648}\right\}$$

$$= \frac{23WL^3}{648EI}$$

From figure, K_{EA} is equal to δ_E.

$$\text{Therefore } \delta_E = \frac{23WL^3}{648EI} \quad (\downarrow)$$

To compute θ_C

From 2nd theorem

$$K_{EC} = \frac{\left(\text{Area of BMD } \bar{X}\right)_{CE}}{EI}$$

$$= \frac{\left(W\,L/3\right)\left(L/6\right)\left(L/12\right)}{EI}$$

$$= \frac{WL^3}{EI}\left(\frac{1}{216}\right)$$

$$= \frac{WL^3}{216EI}$$

$$\delta_c = \delta_E - K_{EC}$$

$$\therefore \delta_c = \frac{23WL^3}{648EI} - \frac{WL^3}{216EI}$$

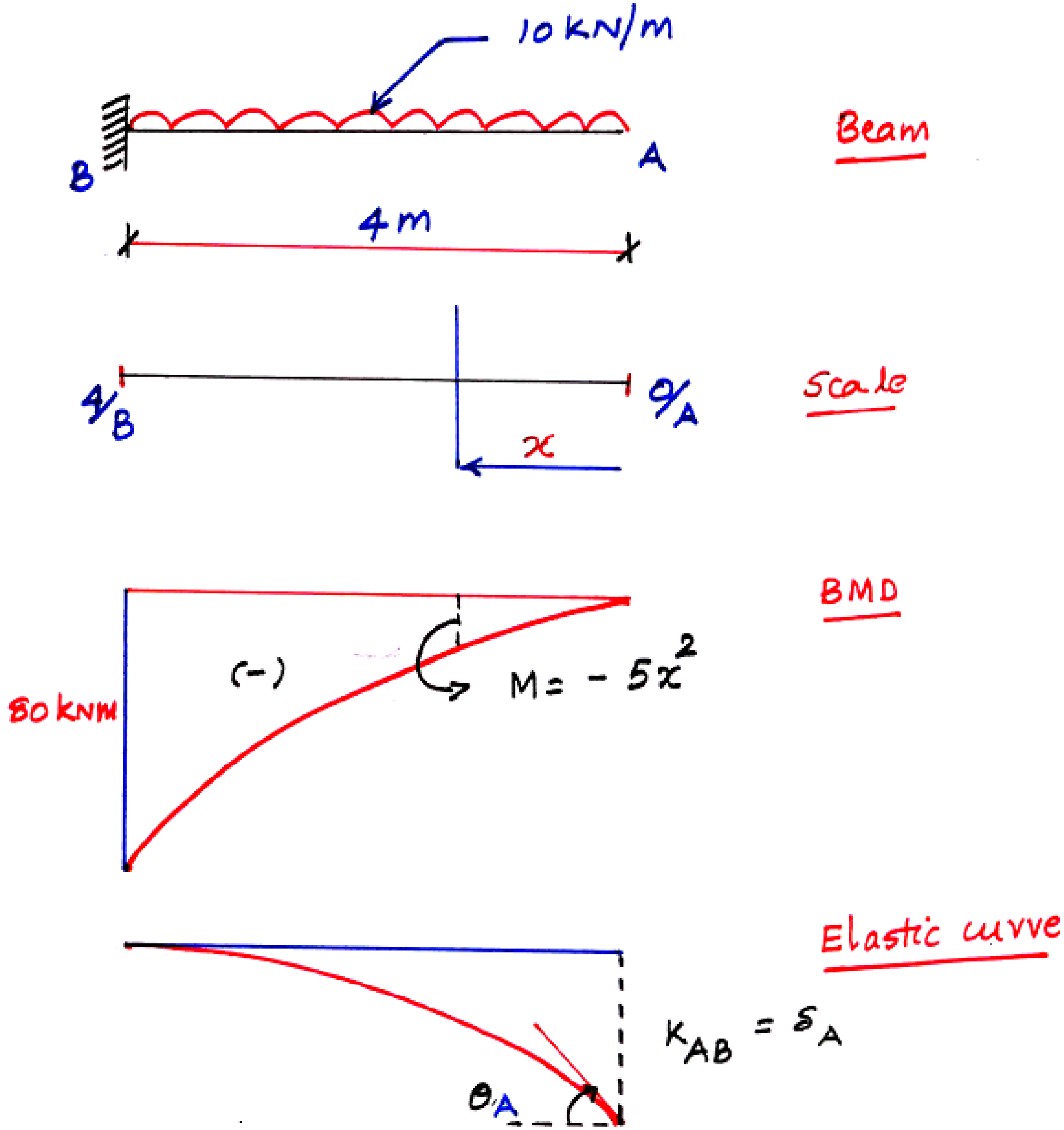
$$= \frac{23WL^3 - 3WL^3}{648EI}$$

$$= \frac{20WL^3}{648EI}$$

$$= \frac{5WL^3}{162EI} \quad (\downarrow)$$

$$= \delta_D = \delta_c = \frac{5WL^3}{162EI} \quad (\downarrow)$$

Problem 2. For the cantilever beam shows in figure, compute deflection and slope at the free end.



Consider a section x-x at a distance x from the free end. The FBD of RHP is taken into account.

(RHP ↶ +) BM @ X-X = M_{X-X} = -10 (x) (x/2) = $-5x^2$

At x = 0; BM @ B = 0

At x = 4m; BM @ A = -5(16) = -80 kNm

The BMD is sketched as shown in figure. Note that it is Hogging Bending Moment. The elastic curve is sketched as shown in figure.

To compute θ_B

For the cantilever beam, at the fixed support, there will be no rotation and hence in this case $\theta_A = 0$. This implies that the tangent drawn to the elastic curve at A will be the same as the beam line.

From I theorem,

$$\theta_{AB} = \theta_A \sim \theta_B = \int_0^4 \frac{M\,dx}{EI}$$

$$= \frac{1}{EI}\int_0^4 \left(-5X^2\right)dx$$

$$= \frac{-5}{EI}\left[x^3/3\right]_0^4$$

$$= \frac{-5}{3EI}(64) = \frac{-320}{3EI}$$

θ_A being zero,

$$\theta_B = \frac{320}{3EI}\ (\circlearrowright)$$

To compute δ_B

From II theorem

$$K_{AB} = \int_0^4 \frac{M\,x\,dx}{EI}$$

$$= \frac{1}{EI}\int_0^4 \left(-5X^2\right)x\,dx$$

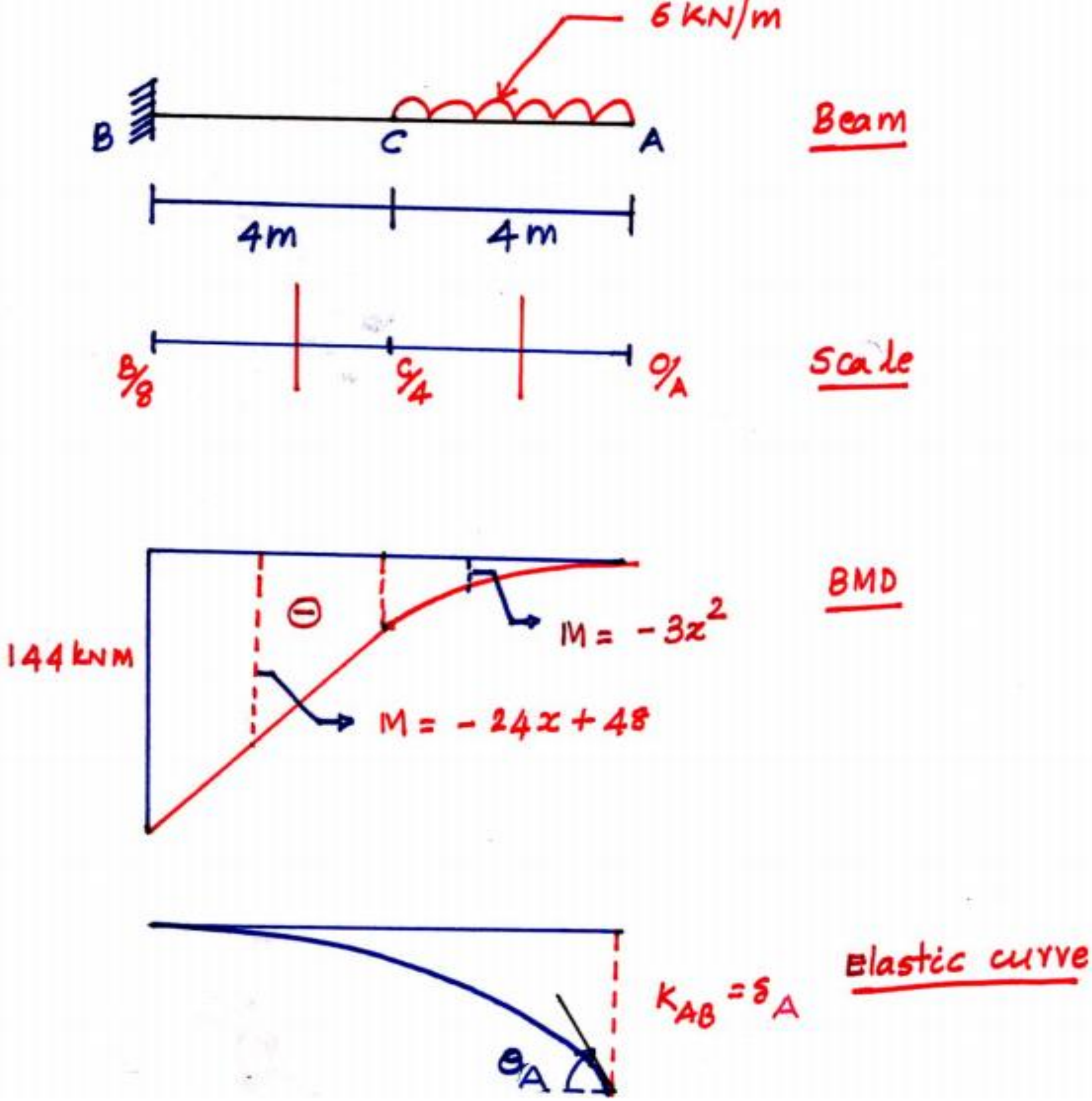
$$= \frac{-5}{EI}\left[x^4/4\right]_0^4 = \frac{-5}{4EI}(256)$$

$$= \frac{-320}{EI}$$

From the elastic curve,

$$K_{AB} = \delta_B = \frac{320}{EI}(\downarrow)$$

Problem 3: Find deflection and slope at the free end for the beam shown in figure by using moment area theorems. Take $EI = 40000\ KNm^{-2}$



Calculations of Bending Moment:

Region AC: Taking RHP ↶ +

Moment at section = $-6x^2/2$

At $x = 0$, BM @ A = 0

$x = 4$m; BM @ C = $-3(16) = -48$kNm

Region CB: ($x = 4$ to $x = 8$)

Taking RHP ↶ +, moment @ section = $-24\,(x-2)$

$= -24x+48;$

At $x = 4$m; BM @ C = $-24(4) + 48 = -48$kNm;

$x = 8$ m BM @ B = -144 kNm;

To compute θ_B:

First moment area theorem is used. For the elastic curve shown in figure. We know that $\theta_A = 0$.

$$\theta_{AB} = \theta_A \sim \theta_B = \int \frac{M\,dx}{EI}$$

$$= \frac{1}{EI}\int_0^4 -3x^2dx + \frac{1}{EI}\int_4^8 (-24x+48)dx$$

$$\theta_A = \frac{-3}{EI}\left[\tfrac{x^3}{3}\right]_0^4 + \frac{1}{EI}\left[-24\tfrac{x^2}{2} + 48x\right]_4^8$$

$$= \frac{-64}{EI} + \frac{1}{EI}\left[-12(64-16)+48(8-4)\right]$$

= -0.0112 Radians

= 0.0112 Radians (↻)

To compute δ_B

$$K_{AB} = \int \frac{Mxdx}{EI}$$

$$= \frac{1}{EI}\int_0^4 -3x^2xdx + \frac{1}{EI}\int_4^8 (-24x+48)xdx$$

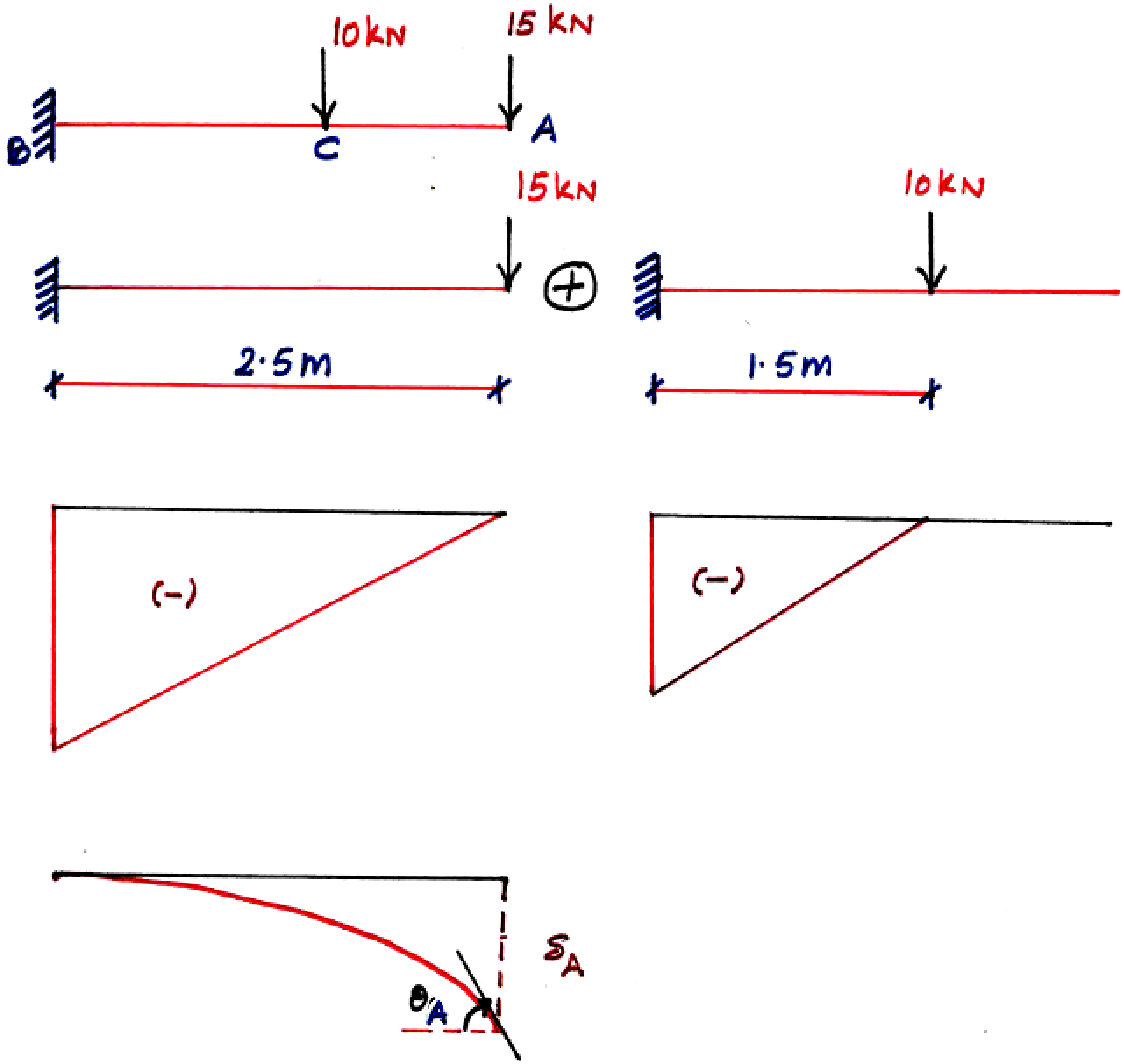
$$= \frac{-3}{EI}\left[\tfrac{x^4}{4}\right]_0^4 + \frac{1}{EI}\left[-24\left(\tfrac{x^3}{3}\right)_4^8 + 48\left(\tfrac{x^2}{2}\right)_4^8\right]$$

$$= \frac{-3}{4EI}[256] + \frac{1}{EI}\left[\frac{-24}{3}(512-64)+24(64-16)\right]$$

$$= \frac{-192}{EI} + \frac{1}{EI}[-3584+1152]$$

$$= \frac{-2624}{EI} = -0.0656m = 0.0656m\downarrow$$

Problem 4: For the cantilever shown in figure, compute deflection and at the points where they are loaded.



To compute θ_B :

$$\theta_{BA} = \theta_B \sim \theta_A = \frac{1}{EI}\left[-\tfrac{1}{2}(2.5)(37.5) - \tfrac{1}{2}(1.5)(15)\right]$$

$$\theta_B = \frac{58.125}{EI}\ (\circlearrowright)$$

$$\theta_C = \frac{1}{EI}\left[-\tfrac{1}{2}(1.5)(37.5+15) - \tfrac{1}{2}(1.5)(15)\right]$$

$$= \frac{50.625}{EI}\ (\circlearrowright)$$

$$\delta_B = -\frac{\tfrac{1}{2}(2.5)(37.5)(\tfrac{2}{3})(2.5)}{EI} - \frac{1}{EI}(\tfrac{1}{2})(1.5)45(1)$$

$$= -\frac{100.625}{EI}$$

$$= -\frac{100.625}{EI}(\downarrow)$$

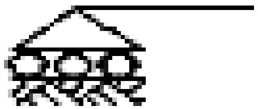
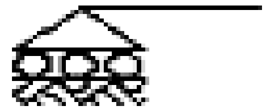
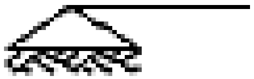
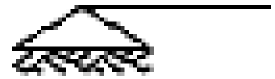
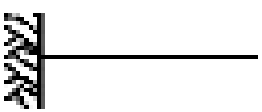
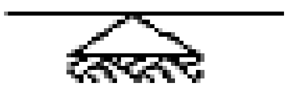
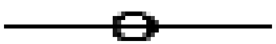
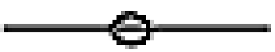
$$\delta_C = \frac{1}{EI}\int \tfrac{1}{2}(1.5)(37.5+15)0.857 + \tfrac{1}{2}(1.5)(45)(1)$$

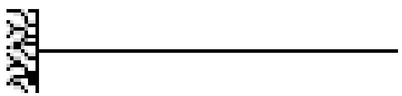
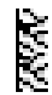
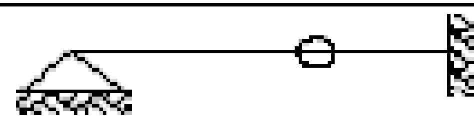
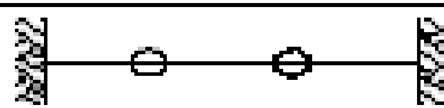
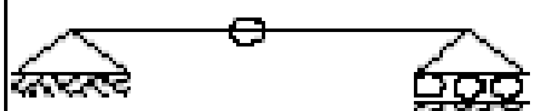
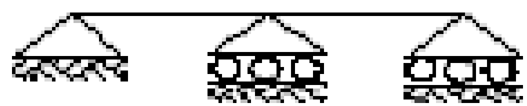
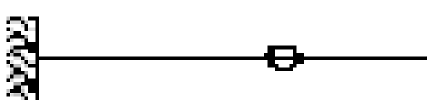
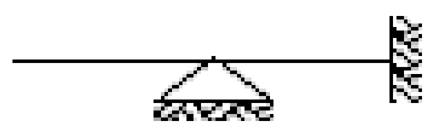
$$\delta_C = \frac{44.99}{EI}(\downarrow)$$

CONJUGATE BEAM METHOD

This is another elegant method for computing deflections and slopes in beams. The principle of the method lies in calculating BM and SF in an imaginary beam called as Conjugate Beam which is loaded with M/EI diagram obtained for real beam. Conjugate Beam is nothing but an imaginary beam which is of the same span as the real beam carrying M/EI diagram of real beam as the load. The SF and BM at any section in the conjugate beam will represent the rotation and deflection at that section in the real beam. Following are the concepts to be used while preparing the Conjugate beam.

- It is of the same span as the real beam.
- The support conditions of Conjugate beam are decided as follows:

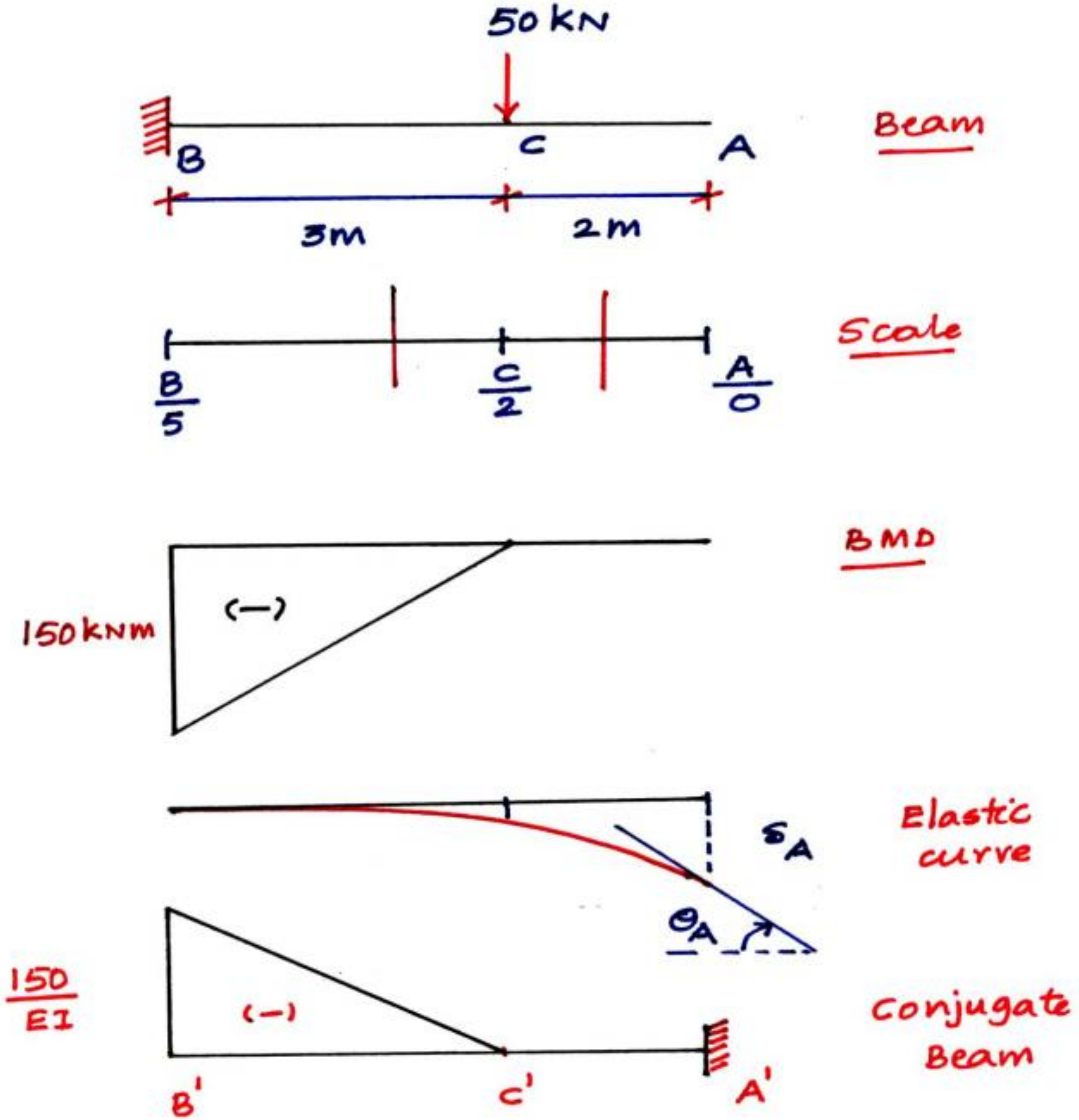
SL NO	SUUPORT IN REAL BEAM	SUUPORT IN CONJUGATE BEAM
1	ROLLER 	ROLLER 
2	HINGE 	HINGE 
3	FIXED 	FREE
4	FREE	FIXED 
5	INTERIOR SUPPORT 	INTERNAL HINGE 
6	INTERNAL HINGE 	INTERIOR SUPPORT 

SL NO	REAL BEAM	CONJUGATE BEAM
1		
2		
3		
4		
5		
6		
7		

Problem 1 : For the Cantilever beam shown in figure, compute deflection and rotation at

(i) the free end

(ii) under the load



Conjugate Beam:

By taking a section @ C′ and considering FBD of LHP,

$$\overset{\uparrow}{+} SF = \sum f_x = \frac{-150}{EI}(3)(\tfrac{1}{2}) = \frac{-225}{EI}$$

$$\text{BM @ C}' = \frac{-150}{EI}(3)(\tfrac{1}{2})(2) = \frac{-450}{EI};$$

Similarly by taking a section at A' and considering FBD of LHP;

$$\text{SF @ A}' = \frac{-225}{EI}$$

$$\text{BM @ A}' = \frac{-225}{EI}(2+2) = \frac{-900}{EI}$$

SF @ a section in Conjugate Beam gives rotation at the same section in Real Beam

BM @ a section in Conjugate Beam gives deflection at the same section in Real Beam

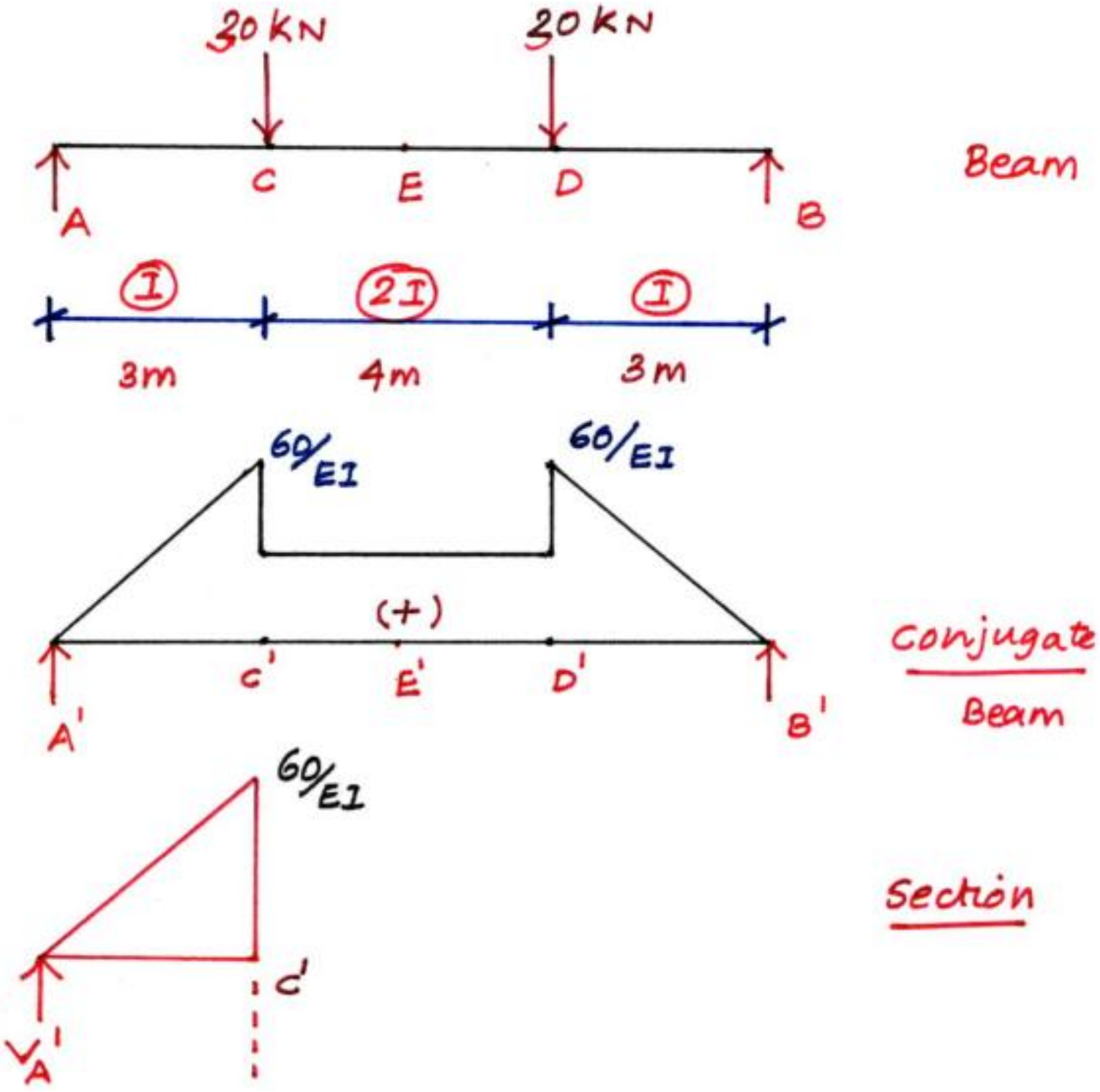
Therefore, Rotation @ C = $\frac{225}{EI}$ (↻)

Deflection @ C= $\frac{450}{EI}(\downarrow)$

Rotation @ A = $\frac{225}{EI}$ (↻)

Deflection @ A = $\frac{900}{EI}(\downarrow)$

Problem 2: For the beam shown in figure, compute deflections under the loaded points. Also compute the maximum deflection. Compute, also the slopes at supports.



For the conjugate beam:

$$V_A' = V_B' = ½[\text{Total load on Conjugate Beam}]$$

$$= ½\left[2(½)(60/_{EI})(3) + 4(30/_{EI})\right]$$

$$= ½\left[\frac{180}{EI} + \frac{120}{EI}\right] = \frac{150}{EI}$$

To compute δ_C :

A section at C' is placed on conjugate beam. Then considering FBD of LHP;

$$\circlearrowleft + \text{ BM @ C'} = \frac{150}{EI}(3) - ½(3)\left(\frac{60}{EI}\right)(1)$$

$$= \frac{450}{EI} - \frac{90}{EI} = \frac{360}{EI}$$

$$\therefore \delta_C = \frac{360}{EI}(\downarrow);$$

To compute δ_E:

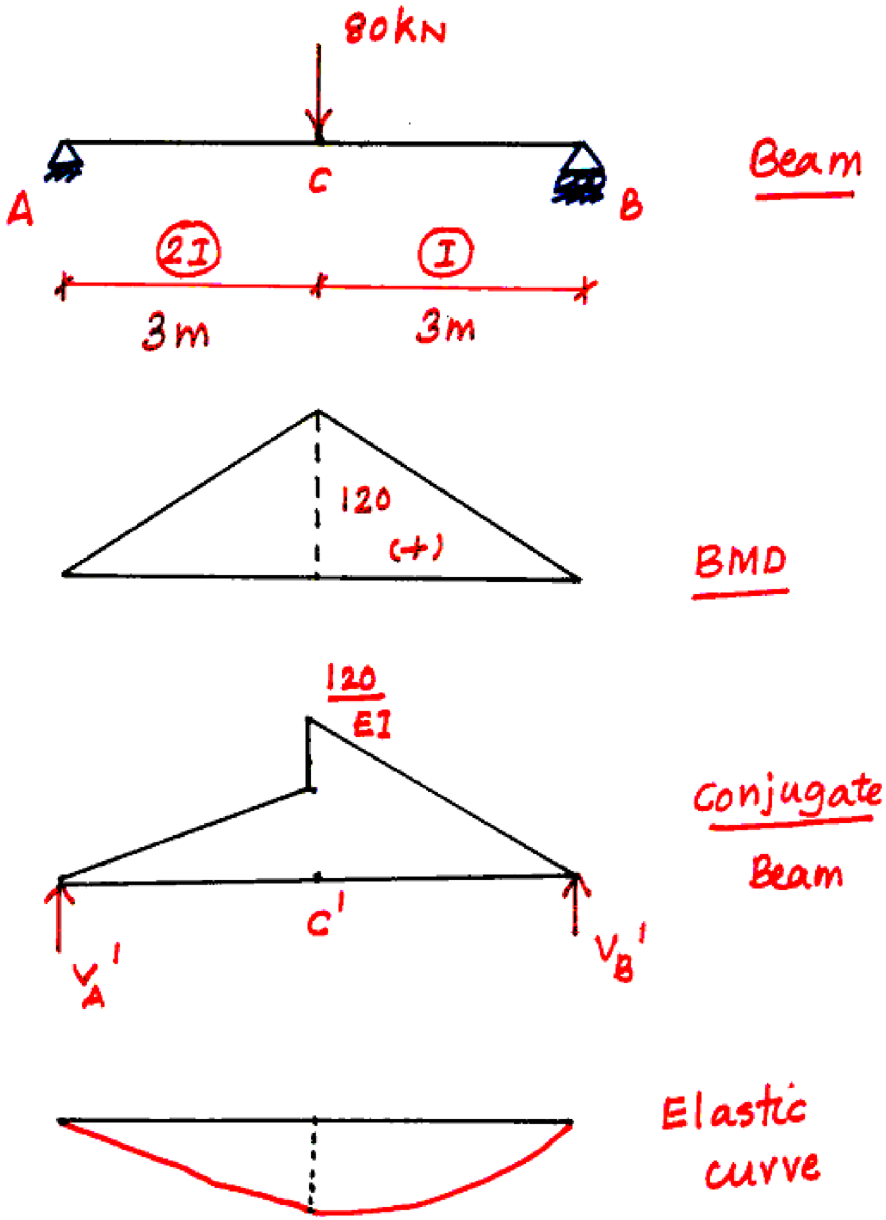
A section @ E' is placed on conjugate beam. Then considering FBD of LHP;

$$+\circlearrowright \text{ BM @ E'} = \frac{150}{EI}(5) - ½(3)\left(\frac{60}{EI}\right)(3) - \frac{30}{EI}(2)(1)$$

$$\text{i.e } \boldsymbol{\delta_E} = \frac{750}{EI} - \frac{270}{EI} - \frac{60}{EI} = \frac{420}{EI}(\downarrow)$$

$$\theta_A = \frac{150}{EI}\ (\circlearrowright) \quad \theta_B = \frac{150}{EI}\ (\circlearrowleft)$$

Problem 3: Compute deflection and slope at the loaded point for the beam shown in figure. Given E = 210 Gpa and I = 120 x 10⁶mm⁴. Also calculate slopes at A and B.



To Compute reactions in Conjugate Beam:

$$\sum fy = 0 \Rightarrow V_A' + V_B' - \left(\frac{1}{2}\right)\left(\frac{60}{EI}\right)(3) - \frac{1}{2}\left(\frac{120}{EI}\right)(3) = 0$$

$$V_A' + V_B' - \frac{90}{EI} - \frac{180}{EI} = 0;$$

$$V_A' + V_B' = \frac{270}{EI}$$

$$\sum m_B = 0 + \circlearrowright \; V_A'(6) - \left(\frac{1}{2}\right)\left(\frac{60}{EI}\right)(3)(4) - \frac{1}{2}\left(\frac{120}{EI}\right)(3)(2$$

$$6V_A' = \frac{360}{EI} + \frac{360}{EI} = \frac{720}{EI}$$

$$V_A' = \frac{120}{EI} \quad ; \quad V_B' = \frac{150}{EI}$$

SF and BM at C' is obtained by placing a section at C' in the conjugate beam.

$$\text{SF @ C'} = \frac{120}{EI} - \frac{1}{2}\left(\frac{60}{EI}\right)(3)$$

$$= \frac{30}{EI}$$

$$+\circlearrowright \text{ BM @ C'} = \frac{120}{EI}(3) - \frac{1}{2}\left(\frac{60}{EI}\right)(3)(1)$$

$$= \frac{360}{EI} - \frac{90}{EI} = \frac{270}{EI}$$

Given $E = 210 \times 10^9 \text{ N/m}^2$

$= 210 \times 10^6 \text{ kN/m}^2$

$I = 120 \times 10^6 \text{ mm}^4$

$= 120 \times 10^6 (10^{-3} \text{ m})^4$

$= 120 \times 10^6 (10^{-12})$

$= 120 \times 10^{-6} \text{ m}^4;$

$EI = 210 \times 10^6 (120 \times 10^{-6}) = 25200 \text{ kNm}^{-2}$

$$\text{Rotation @ C} = \frac{30}{25200} = 1.19 \times 10^{-3} \text{ Radians } (\circlearrowright)$$

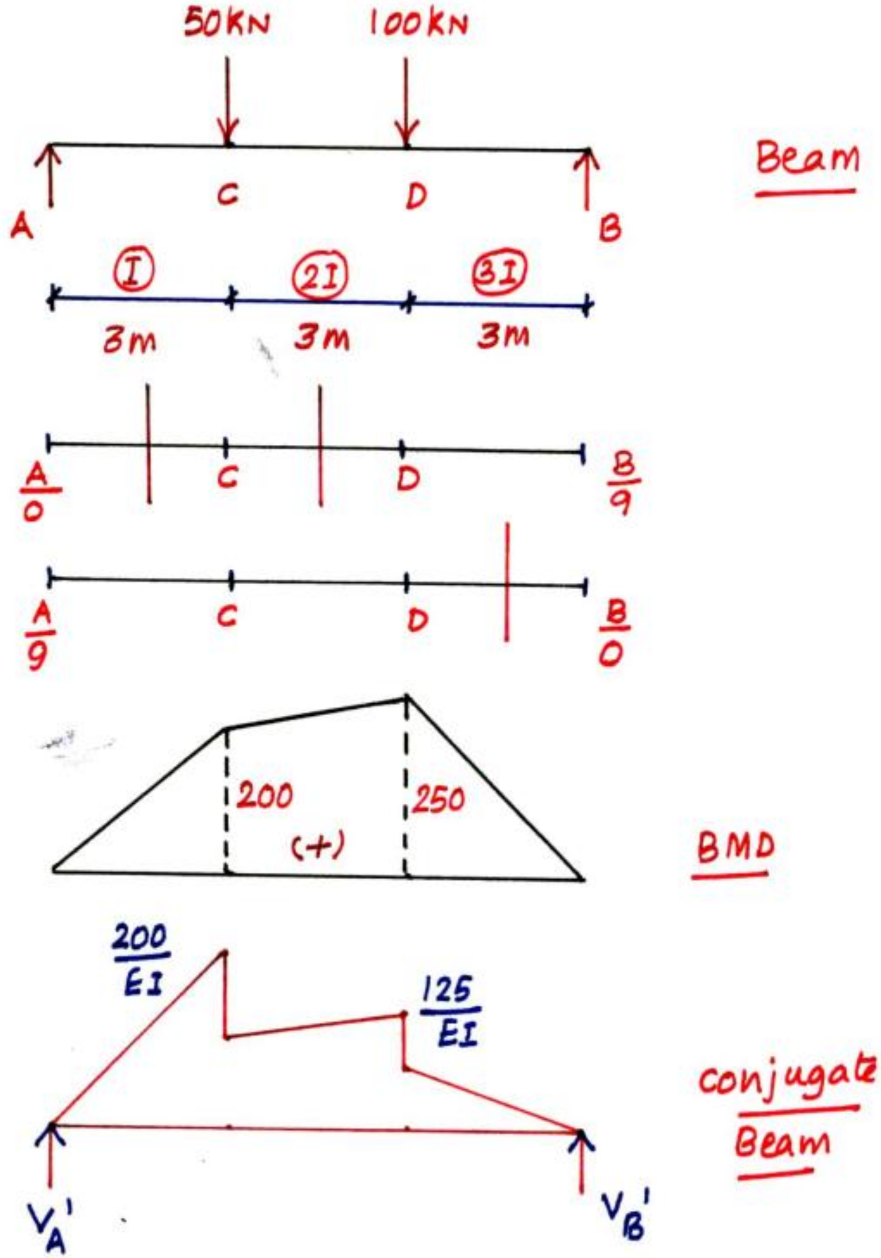
$$\text{Deflection @ C} = \frac{270}{25200} = 0.0107 \text{ m}$$

$= 10.71 \text{ mm } (\downarrow)$

$\theta_A = 4.76 \text{ X } 10^{-3}$ Radians

$\theta_B = 5.95 \text{ X } 10^{-3}$ Radians:

Problem 4: Compute slopes at supports and deflections under loaded points for the beam shown in figure.



To compute reactions and BM in real beam:

$$\uparrow_{+} \sum fy = 0 \Rightarrow V_A + V_B = 150$$

$$+\circlearrowright \sum M_B = 0 \quad 9V_A - 50(6) - 100(3) = 0$$

$$V_A = \frac{600}{9} = 66.67\text{kN} \quad V_B = 83.33\text{kN}$$

BM at (1) – (1) = 66.67 x
At x = 0; BM at A = 0, At x = 3m, BM at C = 200 kNm

BM at (2) – (2) = 66.67 x – 50 (x-3) = 16.67 x + 150
At x = 3m; BM at C = 200 kNm, At x = 6m, BM at D = 250 kNm

BM at (3) – (3) is computed by taking FBD of RHP. Then
BM at (3)-(3) = 83.33 x (x is measured from B)
At x = 0, BM at B = 0, At x = 3m, BM at D = 250 kNm

To compute reactions in conjugate beam:

$$\uparrow_{+} \sum fy = 0 \Rightarrow V_A' + V_B' = \frac{1}{2}(3)\left(\frac{200}{EI}\right) + 3\left(\frac{100}{EI}\right) + \frac{1}{2}(3)\left(\frac{25}{EI}\right) + \frac{1}{2}(3)\left(\frac{83.33}{EI}\right) = \frac{762.5}{EI}$$

$$+\circlearrowright \sum M_B' = 0$$

$$\text{i.e } 9V_A' - \left(\frac{1}{2}\right)(3)\left(\frac{200}{EI}\right)(7) - 3\left(\frac{100}{EI}\right)(4.5) - \left(\frac{1}{2}\right)(3)\left(\frac{25}{EI}\right)(4) - \left(\frac{1}{2}\right)(3)\left(\frac{83.33}{EI}\right)(2) = 0$$

$$9V_A' = \frac{3850}{EI}$$

$$V_A' = \frac{427.77}{EI}$$

$$V_B' = \frac{334.73}{EI}$$

$$\therefore \theta_A = \frac{427.77}{EI} (\circlearrowright) \qquad \theta_B = \frac{334.73}{EI} (\circlearrowleft)$$

To Compute δ_C :

A Section at C' is chosen in the conjugate beam:

$$+\circlearrowright \text{ BM at } C' = \frac{427.77}{EI}(3) - \left(\frac{1}{2}\right)(3)\left(\frac{200}{EI}\right)(1) = \frac{983.31}{EI}$$

$$\therefore \delta_C = \frac{983.31}{EI}(\downarrow)$$

To compute δ_D:

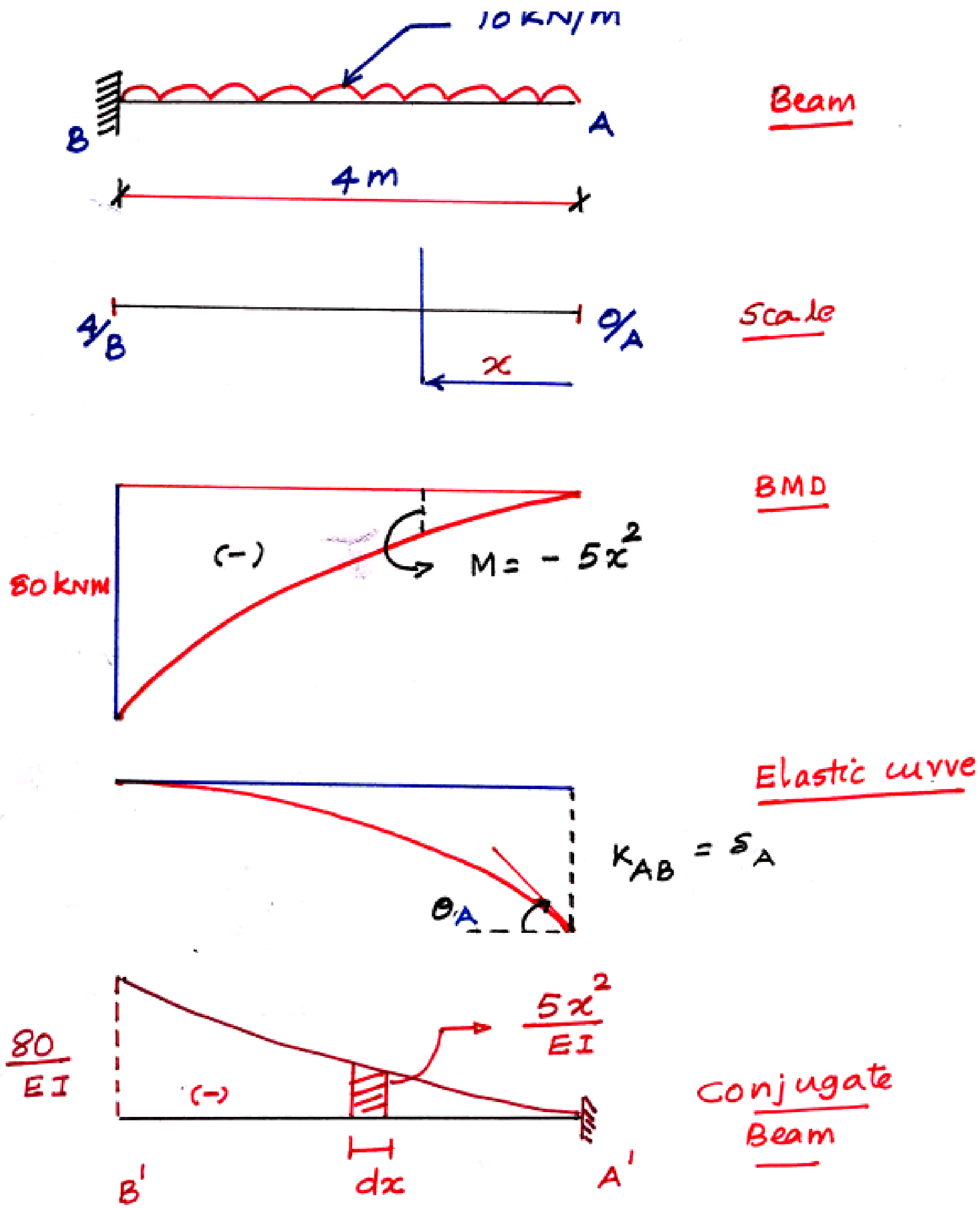
Section at D' is chosen and FBD of RHP is considered.

$$\circlearrowleft + \text{BM at D'} = \frac{334.73}{EI}(3) - \frac{1}{2}(3)\left(\frac{83.33}{EI}\right)(1)$$

$$= \frac{879.19}{EI}$$

$$\delta_D = \frac{879.19}{EI}(\downarrow)$$

Problem 5: Compute to the slope and deflection at the free end for the beam shown in figure.



The Bending moment for the real beam is as shown in the figure. The conjugate beam also is as shown.

Section at A' in the conjugate beam gives

$$\text{SF @ A'} = \int_0^4 \frac{-5x^2}{EI}dx$$

$$= \frac{-5}{EI}\left(x^3/3\right)_0^4 = \frac{-5}{3EI}(64)$$

$$= \frac{-320}{3EI}$$

$$\therefore \quad \theta_A = \frac{320}{3EI}\ (\circlearrowright)$$

$$\text{BM @ A'} = \frac{1}{EI}\int_0^4 -5x^2(x)dx$$

$$= \frac{-5}{EI}\left[\frac{x^4}{4}\right]_0^4 = \frac{-5}{4EI}[256]$$

$$\therefore \delta_A = \frac{320}{EI}(\downarrow)$$

Macaulay's Methods

If the loading conditions change along the span of beam, there is corresponding change in moment equation. This requires that a separate moment equation be written between each change of load point and that two integration be made for each such moment equation. Evaluation of the constants introduced by each integration can become very involved. Fortunately, these complications can be avoided by writing single moment equation in such a way that it becomes continuous for entire length of the beam in spite of the discontinuity of loading.

Note : In Macaulay's method some author's take the help of unit function approximation (i.e. Laplace transform) in order to illustrate this method, however both are essentially the same.

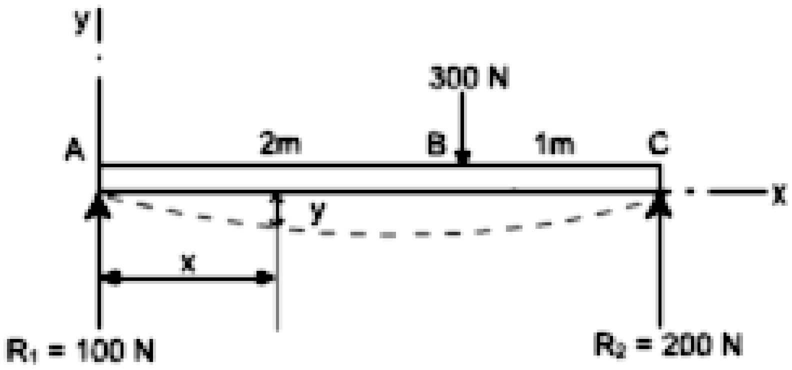
Procedure to solve the problems

(i). After writing down the moment equation which is valid for all values of ‘x' i.e. containing pointed brackets, integrate the moment equation like an ordinary equation.

(ii). While applying the B.C's keep in mind the necessary changes to be made regarding the pointed brackets.

llustrative Examples :

1. A concentrated load of 300 N is applied to the simply supported beam as shown in Fig.Determine the equations of the elastic curve between each change of load point and the maximum deflection in the beam.



Solution : writing the general moment equation for the last portion BC of the loaded beam,

$$EI\frac{d^2y}{dx^2} = M = \left(100x - 300\langle x-2\rangle\right)N.m \qquad ....(1)$$

Integrating twice the above equation to obtain slope and the deflection

$$EI\frac{dy}{dx} = \left(50x^2 - 150\langle x-2\rangle^2 + C_1\right)N.m^2 \qquad ....(2)$$

$$EIy = \left(\frac{50}{3}x^3 - 50\langle x-2\rangle^3 + C_1x + C_2\right)N.m^3 \quad ....(3)$$

To evaluate the two constants of integration. Let us apply the following boundary conditions:

1. At point A where x = 0, the value of deflection y = 0. Substituting these values in Eq. (3) we find C2 = 0.keep in mind that < x -2 >3 is to be neglected for negative values.
2. At the other support where x = 3m, the value of deflection y is also zero. substituting these values in the deflection Eq. (3), we obtain

$$0 = \left(\frac{50}{3}3^3 - 50(3-2)^3 + 3.C_1\right) \text{or } C_1 = -133N.m^2$$

Having determined the constants of integration, let us make use of Eqs. (2) and (3) to rewrite the slope and deflection equations in the conventional form for the two portions.

segment AB ($0 \leq x \leq 2m$)

$$EI\frac{dy}{dx} = \left(50x^2 - 133\right)N.m^2 \qquad ....(4)$$

$$EIy = \left(\frac{50}{3}x^3 - 133x\right)N.m^3 \qquad ....(5)$$

segment BC ($2m \leq x \leq 3m$)

$$EI\frac{dy}{dx} = \left(50x^2 - 150(x-2)^2 - 133x\right)N.m^2 \qquad ....(6)$$

$$EIy = \left(\frac{50}{3}x^3 - 50(x-2)^3 - 133x\right)N.m^3 \qquad ....(7)$$

Continuing the solution, we assume that the maximum deflection will occur in the segment AB. Its location may be found by differentiating Eq. (5) with respect to x and setting the derivative to be equal to zero, or, what amounts to the same thing, setting the slope equation (4) equal to zero and solving for the point of zero slope.

50 x2– 133 = 0 or x = 1.63 m (It may be kept in mind that if the solution of the equation does not yield a value < 2 m then we have to try the other equations which are valid for segment BC)

Since this value of x is valid for segment AB, our assumption that the maximum deflection occurs in this region is correct. Hence, to determine the maximum deflection, we substitute x = 1.63 m in Eq (5), which yields

$$EIy\big|_{max}m = -145N.m^3 \qquad ....(8)$$

The negative value obtained indicates that the deflection y is downward from the x axis.quite usually only the magnitude of the deflection, without regard to sign, is desired; this is denoted by d, the use of y may be reserved to indicate a directed value of deflection.

if E = 30 Gpa and I = 1.9 x 10^6 mm^4 = 1.9 x 10^{-6} m^4 , Eq. (h) becomes

Then

$$y|_{max}m = (30\times10^9)(1.9\times10^{-6})$$

$$= -2.54mm$$

Elastic Stability Of Columns

Introduction:

Structural members which carry compressive loads may be divided into two broad categories depending on their relative lengths and cross-sectional dimensions.

Columns:

Short, thick members are generally termed columns and these usually fail by crushing when the yield stress of the material in compression is exceeded.

Struts:

Long, slender columns are generally termed as struts, they fail by buckling some time before the yield stress in compression is reached. The buckling occurs owing to one the following reasons.

(a). the strut may not be perfectly straight initially.

(b). the load may not be applied exactly along the axis of the Strut.

(c). one part of the material may yield in compression more readily than others owing to some lack of uniformity in the material properties through out the strut.

In all the problems considered so far we have assumed that the deformation to be both progressive with increasing load and simple in form i.e. we assumed that a member in simple tension or compression becomes progressively longer or shorter but remains straight. Under some circumstances however, our assumptions of progressive and simple deformation may no longer hold good and the member become unstable. The term strut and column are widely used, often interchangeably in the context of buckling of slender members.]

At values of load below the buckling load a strut will be in stable equilibrium where the displacement caused by any lateral disturbance will be totally recovered when the disturbance is removed. At the buckling load the strut is said to be in a state of neutral equilibrium, and theoretically it should than be possible to gently deflect the strut into a simple sine wave provided that the amplitude of wave is kept small.

Theoretically, it is possible for struts to achieve a condition of unstable equilibrium with loads exceeding the buckling load, any slight lateral disturbance then causing failure by buckling, this condition is never achieved in practice under static load conditions. Buckling occurs immediately at the point where the buckling load is reached, owing to the reasons stated earlier.

The resistance of any member to bending is determined by its flexural rigidity EI and is The quantity I may be written as $I = Ak^2$,

Where I = area of moment of inertia

A = area of the cross-section

k = radius of gyration.

The load per unit area which the member can withstand is therefore related to k. There will be two principal moments of inertia, if the least of these is taken then the ratio

$$\frac{l}{k} \quad \text{i.e.} \quad \frac{\text{length of member}}{\text{least radius of gyration}}$$

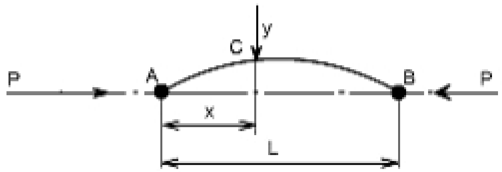
Is called the slenderness ratio. It's numerical value indicates whether the member falls into the class of columns or struts.

Euler's Theory : The struts which fail by buckling can be analyzed by Euler's theory. In the following sections, different cases of the struts have been analyzed.

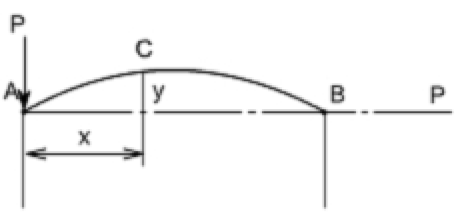
Case A: Strut with pinned ends:

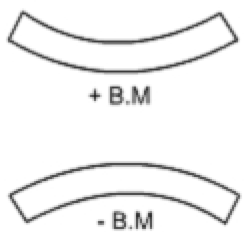
Consider an axially loaded strut, shown below, and is subjected to an axial load 'P' this load 'P' produces a deflection 'y' at a distance 'x' from one end.

Assume that the ends are either pin jointed or rounded so that there is no moment at either end.



Assumption:
The strut is assumed to be initially straight, the end load being applied axially through centroid.





According to sign convention

B. $M|_c = -Py$

Futher,we know that

$$EI\frac{d^2y}{dx^2} = M$$

$$EI\frac{d^2y}{dx^2} = -P.y = M$$

In this equation 'M' is not a function 'x'. Therefore this equation can not be integrated directly as has been done in the case of deflection of beams by integration method.

Thus,

$$EI\frac{d^2y}{dx^2} + Py = 0$$

Though this equation is in 'y' but we can't say at this stage where the deflection would be maximum or minimum.

So the above differential equation can be arranged in the following form

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = 0$$

Let us define a operator

D = d/dx

$(D^2 + n^2)\, y = 0$ where $n^2 = P/EI$

This is a second order differential equation which has a solution of the form consisting of complimentary function and particular integral but for the time being we are interested in the complementary solution only[in this P.I = 0; since the R.H.S of Diff. equation = 0]

Thus $y = A \cos(nx) + B \sin(nx)$

Where A and B are some constants.

Therefore $y = A \cos\sqrt{\frac{P}{EI}}\, x + B \sin\sqrt{\frac{P}{EI}}\, x$

In order to evaluate the constants A and B let us apply the boundary conditions,

(i) at $x = 0$; $y = 0$

(ii) at $x = L$; $y = 0$

Applying the first boundary condition yields $A = 0$.

Applying the second boundary condition gives

$$B \sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

Thus either $B = 0$, or $\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$

if B=0,that y0 for all values of x hence the strut has not buckled yet.Therefore,the solution requir

$$\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0 \quad \text{or} \quad \left(L\sqrt{\frac{P}{EI}}\right) = \pi \text{ or } nL = \pi$$

$$\text{or} \quad \sqrt{\frac{P}{EI}} = \frac{\pi}{L} \quad \text{or} \quad P = \frac{\pi^2 EI}{L^2}$$

From the above relationship the least value of P which will cause the strut to buckle, and it is called the " **Euler Crippling Load** " P_e from which w obtain.

Limitations of Euler's Theory :

In practice the ideal conditions are never [i.e. the strut is initially straight and the end load being applied axially through centroid] reached. There is always some eccentricity and initial curvature present. These factors needs to be accommodated in the required formula's.

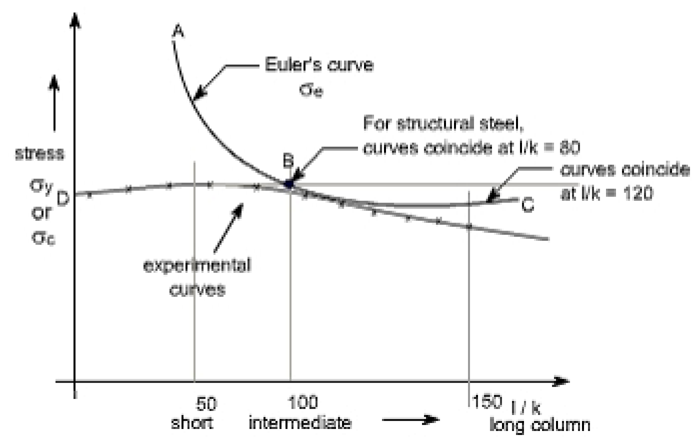
It is realized that, due to the above mentioned imperfections the strut will suffer a deflection which increases with load and consequently a bending moment is introduced which causes failure before the Euler's load is reached. Infact failure is by stress rather than by buckling and the deviation from the Euler value is more marked as the slenderness-ratio l/k is reduced. For values of l/k < 120 approx, the error in applying the Euler theory is too great to allow of its use. The stress to cause buckling from the Euler formula for the pin ended strut is

$$\text{Euler's stress, } \sigma_e = \frac{P_e}{A} = \frac{\pi^2 EI}{Al^2}$$

But, $I = Ak^2$

$$\sigma_e = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$$

A plot of s_e versus l / k ratio is shown by the curve ABC.



UNIT 5

ANALYSIS OF STRESSESS IN TWO DIMENSIONS

4.1 DERIVATION OF GENERAL EQUATIONS

Consider the complex stress system in Figure 4.1 acting on an element of material. The stresses σ_x and σ_y may be compressive or tensile and may be the result of direct forces or bending. The shear stresses may be as shown or completely reversed and occur as a result of either shear forces or torsion. Since the applied and complementary shear stresses are of equal value on the x and y planes, they are both given the symbol, τ_{xy}.

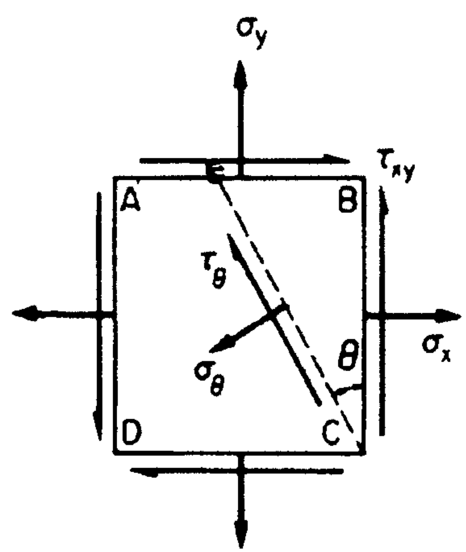


Fig 4.1 : Two-dimensional complex stress system.

The diagram thus represents a complete stress system for any condition of applied load in two dimensions. Consider the rectangular element of unit depth shown in Figure 4.1 . subjected to a system of two direct stresses and shear stresses.

For equilibrium of the portion EBC (Figure 4.1), redrawn in Figure 4.2:

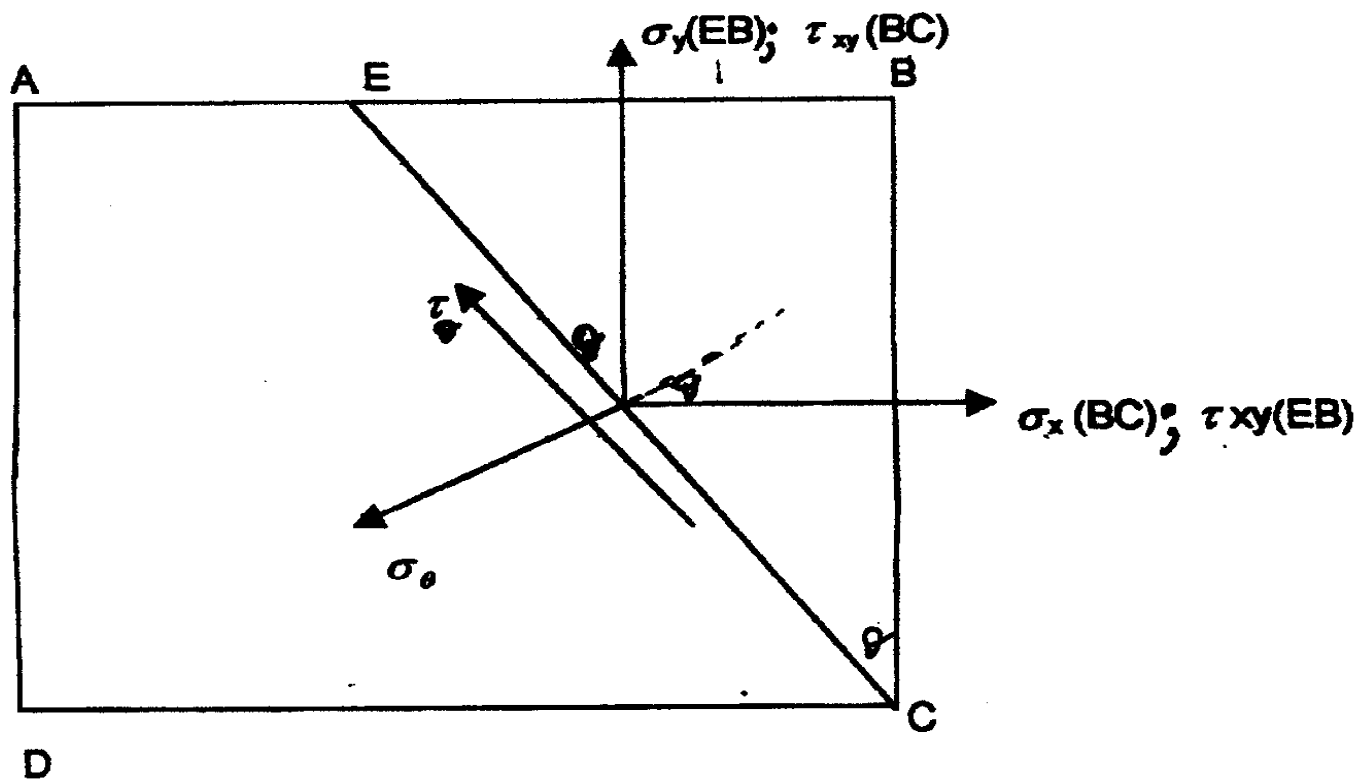
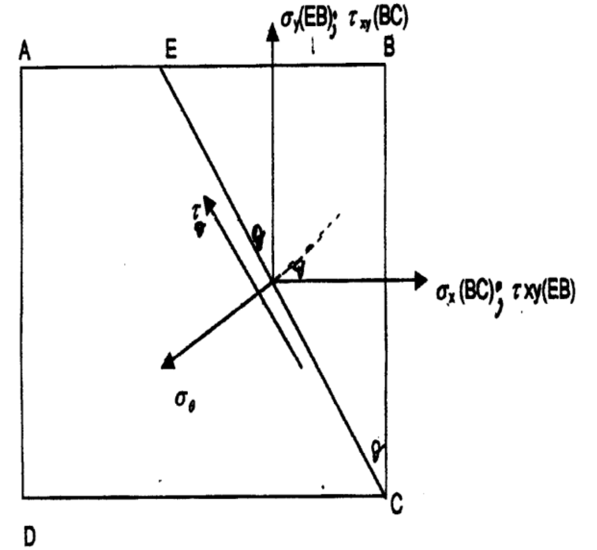


Figure 4.2: Forces Acting on Element EBC

Resolving perpendicular to EC:



$$\sigma_\theta \times 1 \times EC = \sigma_x \times BC \times 1 \times \cos\theta$$

$$+ \sigma_y \times EB \times 1 \times \sin\theta$$

$$+ \tau_{xy} \times 1 \times EB \times \cos\theta$$

$$+ \tau_{xy} \times 1 \times BC \times \sin\theta$$

Note that $EB = EC \sin\theta$ and $BC = EC \cos\theta$

$$\sigma_\theta \times EC = \sigma_x \times EC \cos^2\theta + \sigma_y \times EC \sin^2\theta$$

$$+ \tau_{xy} \times EC \times \sin\theta \cos\theta$$

$$+ \tau_{xy} \times EC \sin\theta \cos\theta$$

$$\sigma_\theta = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

Recall that : $\cos^2\theta = (1 + \cos 2\theta)/2$, $\sin^2\theta = (1 - \cos 2\theta)/2$ and

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

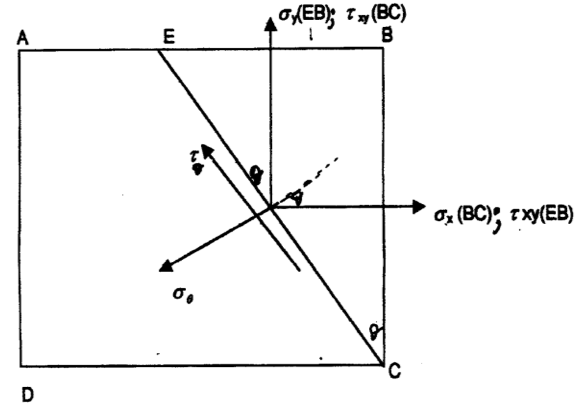
$$\sigma_\theta = \sigma_x/2\ (1 + \cos 2\theta) + \sigma_y/2\ (1 - \cos 2\theta) + \tau_{xy} \sin 2\theta$$

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy} \sin 2\theta \quad \dots\dots\dots\dots\dots\dots\dots \quad (4.1)$$

Resolving parallel to EC:

$$\tau_\theta \times 1 \times EC = \sigma_x \times BC \times 1 \times \sin\theta + \sigma_y \times EB \times 1 \times \cos\theta$$

$$+ \tau_{xy} \times 1 \times EB \times \sin\theta + \tau_{xy} \times 1 \times BC \times \cos\theta$$



Derivation of General Equation Concluded

τ_θ x EC = σ_x x EC $\sin\theta \cos\theta$ σ_y x EC $\sin\theta$ $\cos\theta$ +

τ_{xy} x EC x $\sin^2\theta$ - τ_{xy} x EC $\cos^2\theta$

$\tau_\theta = \sigma_x \sin\theta \cos\theta - \sigma_y \sin\theta \cos\theta + \tau_{xy} \sin^2\theta - \tau_{xy} \cos^2\theta$

Recall that $\sin 2\theta = 2 \sin\theta \cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$\tau_\theta = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (4.2)$$

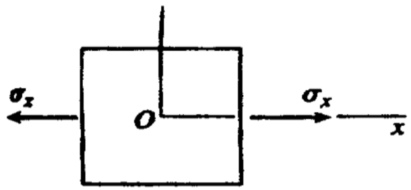
SPECIAL CASES OF PLANE STRESS

The general case of plane stress reduces to simpler states of stress under special conditions:

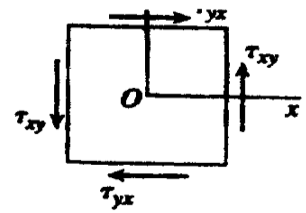
4.1.1 **Uniaxial Stress**: This is the situation where all the stresses acting on the xy element are zero except for the normal stress σ_x, then the element is in uniaxial stress. The corresponding transformation equations, obtained by setting σ_y and τ_{xy} equal to zero in the Equations 4.1 and 4.2 above:

$$\sigma_\theta = \frac{\sigma_x}{2}(1 + \cos 2\theta), \qquad \tau_\theta = \frac{\sigma_x}{2}\sin 2\theta$$

Special Cases of Plane Stress Contd.



Element in uniaxial stress



Element in pure shear

4.2.2 Pure Shear: The transformation equations are obtained by substituting $\sigma_x = 0$ and $\sigma_y = 0$ into Equations 4.1 and 4.

$$\sigma_\theta = \tau_{xy} \sin 2\theta \qquad \tau_\theta = \tau_{xy} \cos 2\theta$$

4.2.3 Biaxial Stress: The xy element is subjected to normal stresses in both x and y directions but without any shear stresses. τ_{xy} is merely dropped from the general equations to obtain:

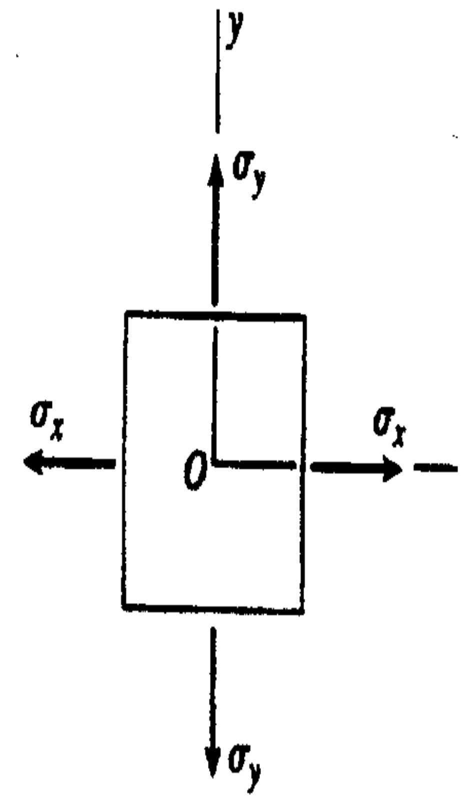
$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta \qquad \tau_\theta = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta$$

The maximum direct stress will equal σ_x or σ_y, whichever is the greater, when $\theta = 0$ or 90°.

Maximum Shear Stress

The maximum shear stress in the plane of the applied stresses occurs when $\theta = 45^\circ$, i.e.

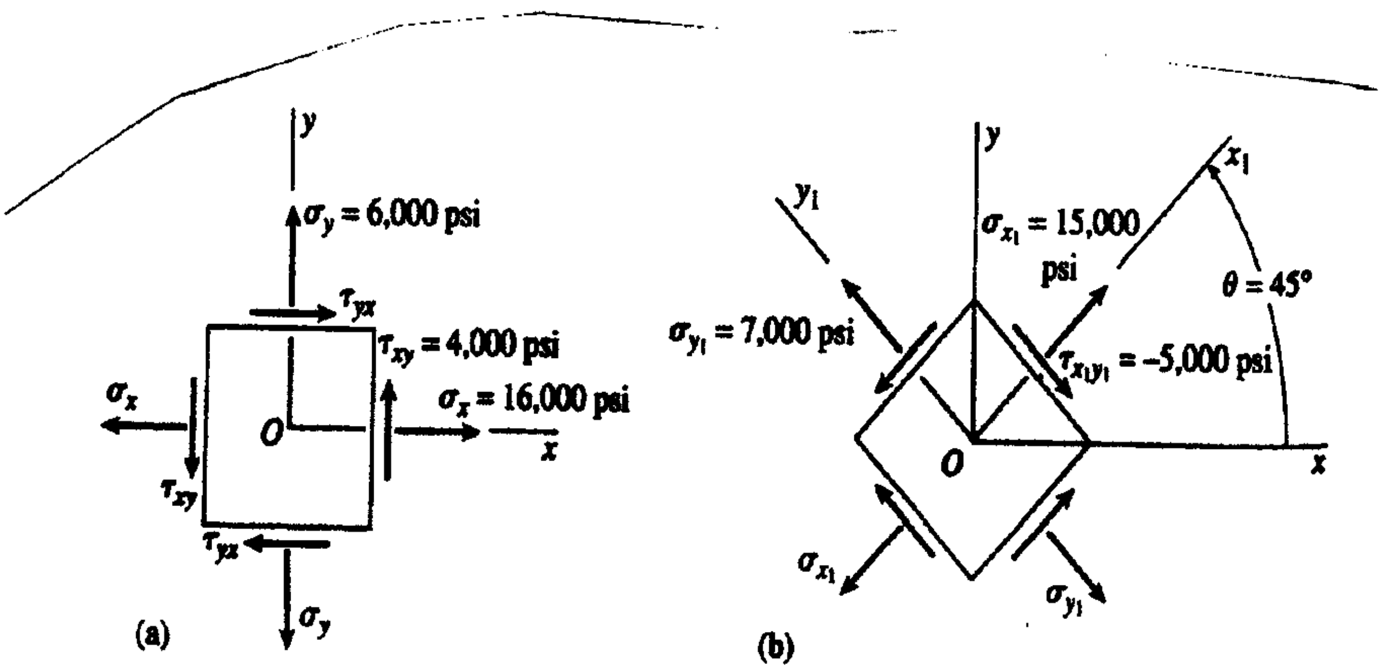
$$\tau_{max} = \frac{1}{2}(\sigma_x - \sigma_y) \qquad \ldots\ldots\ldots\ldots (4.3)$$



Element in biaxial stress

Example

Example: An element in plane stress is subjected to stresses $\sigma_x = 16{,}000\ N/mm^2$, $\sigma_y = 6000\ N/mm^2$, and $\tau_{xy} = 4000\ N/mm^2$. Determine the stresses acting on an element inclined at an angle of $\theta = 45°$.



Solution

Solution: To obtain the stresses on an inclined element, use equations (4.1) and (4.2)

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_\theta = \frac{16{,}000 + 6{,}000}{2} + \frac{16{,}000 - 6000}{2}\cos 90^\circ + 4000\sin 90 = 15{,}000 N/mm^2$$

$$\tau_\theta = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta = \frac{16000 - 6000}{2}\sin 90 + 4000\cos 90^\circ = 5000 N/mm^2$$

Principal Stresses and Maximum Shear Stresses

4.3 PRINCIPAL STRESSES AND MAXIMUM SHEAR STRESSES

The maximum and minimum stresses which occur on any plane in the material can now be determined as follows:

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \quad - - \quad . \quad (4.1)$$

For σ_θ to be a maximum or minimum, $d\sigma_\theta / d\theta = 0$

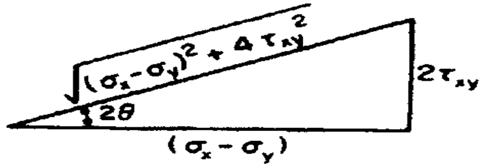
$$\frac{d\sigma_\theta}{d_\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0 \qquad \div \cos 2\theta$$

$$\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)} \quad \dots\dots\dots\dots\dots\dots \quad (4.4)$$

From Figure below:

$$\sin 2\theta = \frac{2\tau_{xy}}{\sqrt{\left[(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}\right]}}$$

$$\cos 2\theta = \frac{(\sigma_x - \sigma_y)}{\sqrt{\left[(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}\right]}}$$



Principal Stresses and Maximum Shear Stresses Contd.

The solution of equation 4.4 yields two values of 2θ separated by 180^o, i.e. two values of θ separated by 90^o. Thus the two principal stresses occur on mutually perpendicular planes termed principal planes,

Substituting in equation 4.1:

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \quad \frac{(\sigma_x - \sigma_y)}{\sqrt{\left|(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}\right|}} + \tau_{xy} \quad \frac{2\tau_{xy}}{\sqrt{\left|(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}\right|}}$$

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} \pm \frac{(\sigma_x - \sigma_y)^2}{2\sqrt{\left|(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}\right|}} + \frac{2\tau^2_{xy}}{\sqrt{\left|(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}\right|}}$$

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2} \frac{(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}}{\sqrt{\left|(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}\right|}}$$

Shear Stresses at Principal Planes are Zero

$$\sigma_1 \text{ or } \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2}\sqrt{\left|(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}\right|} \quad \ldots\ldots\ldots (4.5)$$

These are termed the principal stresses of the system. By substitution for θ from equation 4.4, into the shear stress expression (equation 4.2):

$$\tau_\theta = \frac{\sigma_x - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots. (4.2)$$

$$\tau_\theta = \frac{\sigma_x - \sigma_y}{2} \frac{2\tau_{xy}}{\sqrt{\left|(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}\right|}} - \tau_{xy}\frac{(\sigma_x - \sigma_y)}{\sqrt{\left|(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}\right|}}$$

$$\tau_\theta = \frac{\tau_{xy}(\sigma_x - \sigma_y)}{\sqrt{\left|(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}\right|}} - \frac{\tau_{xy}(\sigma_x - \sigma_y)}{\sqrt{\left|(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}\right|}} = 0$$

Principal Planes and Stresses Contd.

Thus at principal planes, $\tau_\theta = 0$. Shear stresses do not occur at the principal planes.

The complex stress system of Figure 4.1 can now be reduced to the equivalent system of principal stresses shown in Figure 4.2 below.

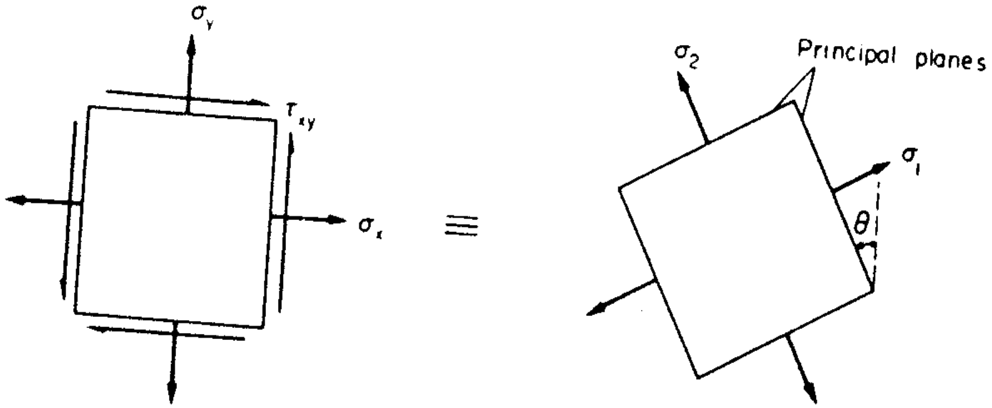


Figure 4.3: Principal planes and stresses

Equation For Maximum Shear Stress

From equation 4.3, the maximum shear stress present in the system is given by:

$$\tau_{max} = \frac{1}{2}(\sigma_x - \sigma_y) = \frac{1}{2}\sqrt{\left|(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy}\right|}$$

and this occurs on planes at 45^0 to the principal planes.

Note: This result could have been obtained using a similar procedure to that used for determining the principal stresses, i.e. by differentiating expression 4.2, equating to zero and substituting the resulting expression for θ

4.4 PRINCIPAL PLANE INCLINATION IN TERMS OF THE ASSOCIATED PRINCIPAL STRESS

It has been stated in the previous section that expression (4.4), namely

$$\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

yields two values of θ, i.e. the inclination of the two principal planes on which the principal stresses σ_1 or σ_2. It is uncertain, however, which stress acts on which plane unless eqn. (4.1) is used, substituting *one* value of θ obtained from eqn. (4.4) and observing which one of the two principal stresses is obtained. The following alternative solution is therefore to be preferred.

PRINCIPAL PLANE INCLINATION CONTD.

- Consider once again the equilibrium of a triangular block of material of unit depth (Fig. 4.3); this time *EC* is a principal plane on which a principal stress acts, and the shear stress is zero (from the property of principal planes).

PRINCIPAL PLANE INCLINATION CONTD.

Resolving forces horizontally,

$$(\sigma_x x\ BC \times 1) + (\tau_{xy} \times EB \times 1) = (\sigma_p \times EC \times 1) \cos\theta$$

$$\sigma_x EC \cos\theta + \tau_{xy} \times EC \sin\theta = \sigma_p \times EC \cos\theta$$

$$\sigma_x + \tau_{xy} \tan\theta = \sigma_p$$

$$\tan\theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}} \quad \ldots \ (4.7)$$

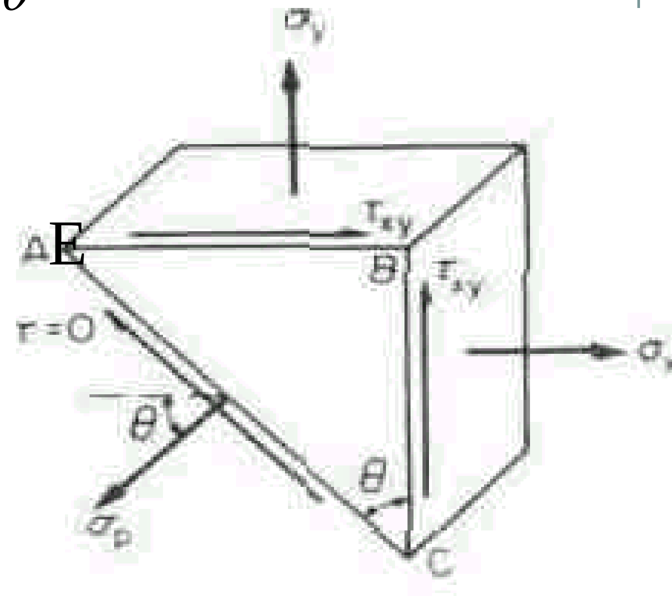


Fig. 13.8.

PRINCIPAL PLANE INCLINATION CONTD.

- Thus we have an equation for the inclination of the principal planes *in terms of the* principal stress. If, therefore, the principal stresses are determined and substituted in the above equation, each will give the corresponding angle of the plane on which it acts and there can then be no confusion.

PRINCIPAL PLANE INCLINATION CONTD.

- The above formula has been derived with two tensile direct stresses and a shear stress system, as shown in the figure; should any of these be reversed in action, then the appropriate minus sign must be inserted in the equation.

Graphical Solution Using the Mohr's Stress Circle

4.5. GRAPHICAL SOLUTION-MOHR'S STRESS CIRCLE

Consider the complex stress system of Figure below. As stated previously this represents a complete stress system for any condition of applied load in two dimensions. In order to find graphically the direct stress σ_p and shear stress $\theta\tau$ on any plane inclined at θ to the plane on which σ_x acts, proceed as follows:

(1) Label the block *ABCD.*

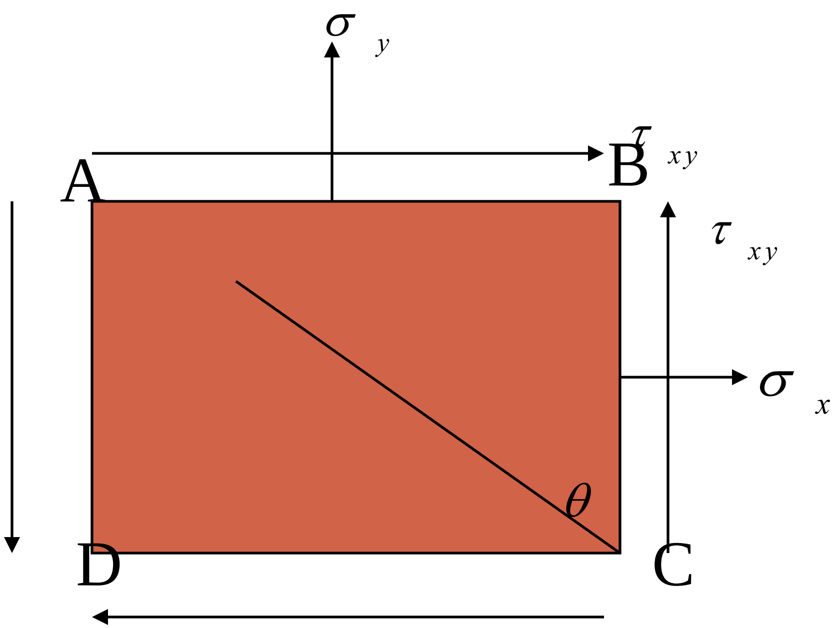
(2) Set up axes for direct stress (as abscissa) and shear stress (as ordinate)

(3) Plot the stresses acting on two *adjacent* faces, e.g. *AB* and *BC*, using the following sign conventions:

Mohr's Circle Contd.

- *Direct stresses:* tensile, positive; compressive, negative;
- *Shear stresses:* tending to turn block clockwise, positive; tending to turn block
  counterclockwise, negative.
- This gives two points on the graph which may then be labeled *AB* and *BC* respectively to denote stresses on these planes

Mohr's Circle Contd.



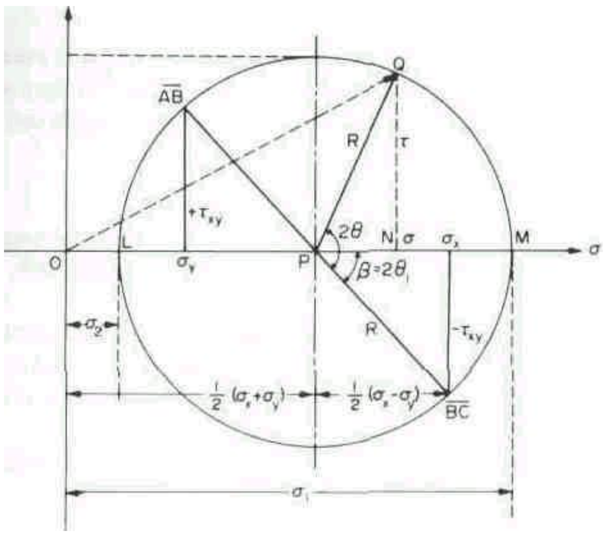
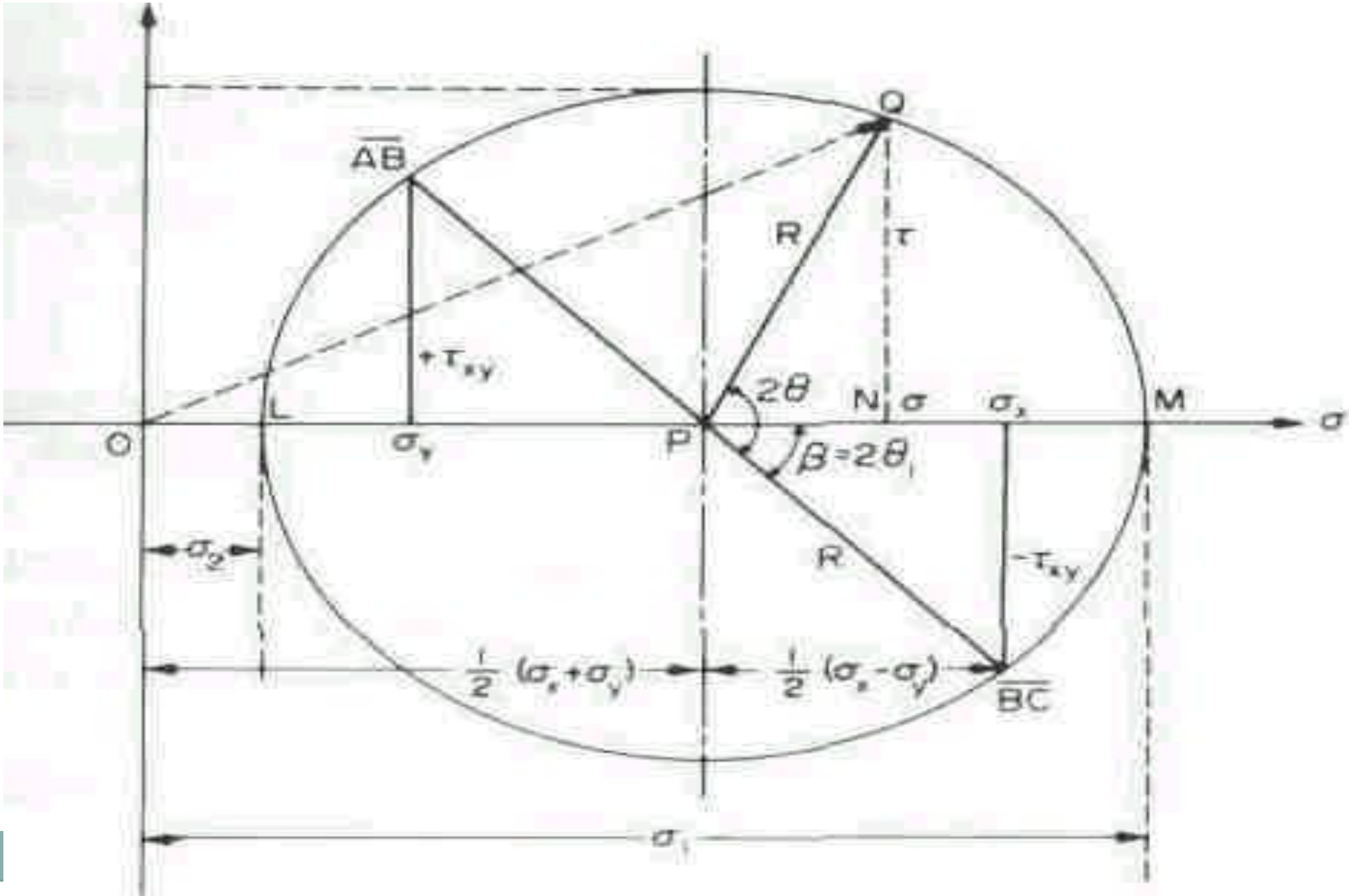


Fig. 4.5 **Mohr's stress circle.**

(4) Join *AB* and *BC*.

(5) The point *P* where this line cuts the *a* axis is then the centre of Mohr's circle, and the line is the diameter; therefore the circle can now be drawn. Every point on the circumference of the circle then represents a state of stress on some **plane** through C.

Mohr's stress circle.



Proof

Consider any point Q on the circumference of the circle, such that PQ makes an angle 2θ with BC, and drop a perpendicular from Q to meet the a axis at N.

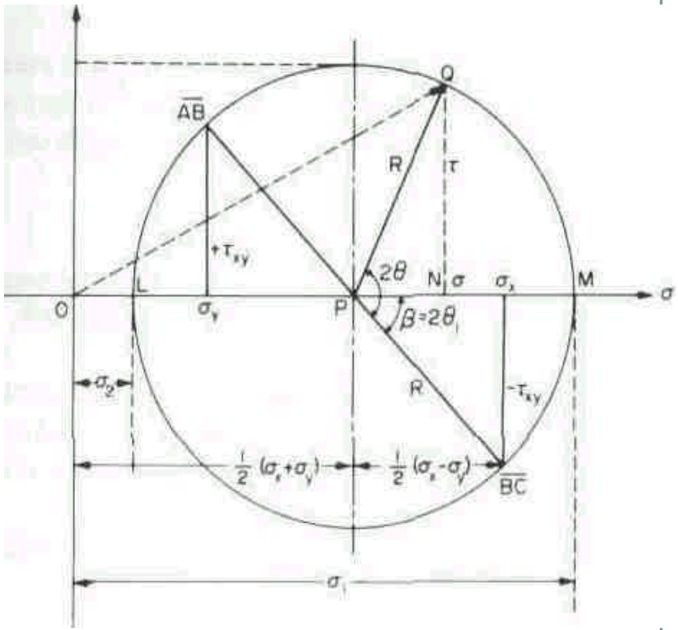
***Coordinates of** Q:*

$$ON = OP + PN = \frac{1}{2}(\sigma_x + \sigma_y) + R\cos(2\theta - \beta)$$

$$\frac{1}{2}(\sigma_x + \sigma_y) + R\cos 2\theta \cos\beta + R\sin 2\theta \sin\beta$$

$$R\cos\beta = \frac{1}{2}(\sigma_x - \sigma_y) \quad and \quad R\sin\beta = \tau_{xy}$$

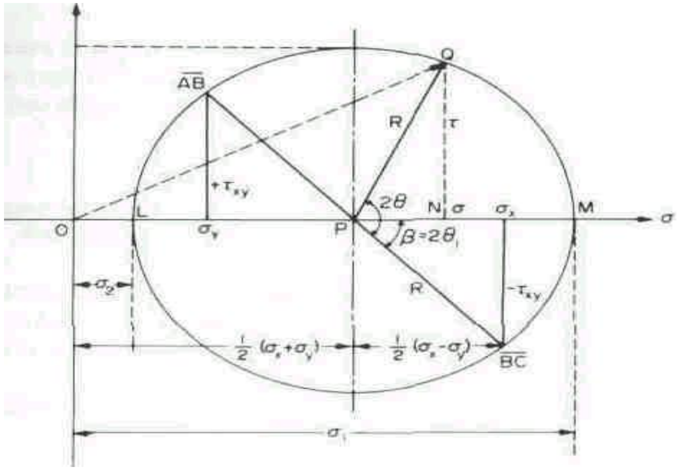
$$ON = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$



Proof Contd.

On inspection this is seen to be eqn. (4.1) for the direct stress σ_θ on the plane inclined at θ to *BC* in the figure for the two-dimensional complex system.

Similarly,



$$QN \quad \sin(2\theta - \beta)$$

$$= R \sin 2\theta \cos\beta - R\cos 2\theta \sin\beta$$

$$= \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

Again, on inspection this is seen to be eqn. (4.2) for the shear stress τ_θ on the plane inclined at θ to *BC*.

Note

Thus the coordinates of *Q* are the normal and shear stresses on a plane inclined at θ to BC in the original stress system.

N.B. - Single angle *BCPQ* is 2θ on Mohr's circle and not θ, it is evident that angles *are doubled on Mohr's circle.* This is the only difference, however, as they are measured in the same direction and from the same plane in both figures (in this case counterclockwise from

~*BC).*

Further Notes on Mohr's Circle

Further points to note are:

(1) The direct stress is a maximum when *Q* is at *M*, i.e. *OM* is the length representing the maximum principal stress σ_1 and $2\theta_1$ gives the angle of the plane θ_1 from *BC*. Similarly, *OL is* the other principal stress.

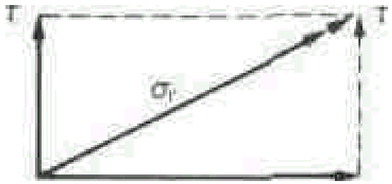
(2) The maximum shear stress is given by the highest point on the circle and is represented by the radius of the circle. This follows since shear stresses and complementary shear stresses have the same value; *therefore the centre of the circle will always lie on the* σ_1 *axis midway between* σ_x *and* σ_y .

(3) From the above point the direct stress on the plane of maximum shear must be midway between σ_x *and* σ_y .

Further Notes on Mohr Circle Contd.

(4) The shear stress on the principal planes is zero.

(5) Since the resultant of two stresses at 90° can be found from the parallelogram of vectors as the diagonal, as shown in Figure below, the resultant stress on the plane at θ to BC is given by OQ on Mohr's circle.



Resultant stress σ_r on any plane.

Preference of Mohr Circle

- The graphical method of solution of complex stress problems using Mohr's circle is a very powerful technique since all the information relating to any plane within the stressed element is contained in the single construction.
- It thus provides a convenient and rapid means of solution which is less prone to arithmetical errors and is highly recommended.